On Energy Efficiency and Spectral Efficiency Joint Optimization of Ultra Dense Networks

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Abstract—Deploying ultra dense networks (UDNs) is a major trend in the evolution of cellular networks, where the number of base stations (BSs) may exceed the number of users. In this paper, we investigate energy and spectral efficient frequency reuse strategies in hexagonally deployed cells, where BS sleeping is allowed for a cell without active users. Toward this goal, we derive the closed form expression of the energy efficiency (EE) and spectral efficiency (SE), and find the optimal frequency reuse strategies that maximize the EE and SE, respectively. Our analyses show that full frequency reuse is SE-optimal for cellular networks with all ratios of BS to user density, but is EE-optimal only when the BS-user density ratio exceeds a threshold (i.e. in UDNs), while using orthogonal frequency band among adjacent cells is EE-optimal when the ratio of density is less than the threshold. The normalized EE gain of full frequency reuse in UDNs increases with the number of BSs. Simulation results validate our analysis and illustrate the EE gain with typical BS-user density ratios.

I. INTRODUCTION

Energy efficiency (EE) is a major design goal for the fifth-generation (5G) cellular networks [1]. To support the 1000-fold higher network throughput, deploying ultra dense networks (UDNs) has become a promising approach, where the number of low power base stations (BSs) may exceed the number of mobile stations (MSs) [2–4].

Under the assumption of heavy load and uniform traffic, in traditional cellular networks all the BSs are always operated. In such networks, full frequency reuse is optimal to maximize the spectral efficiency (SE). In a full frequency reused UDN with all BSs activated, the BS density can be optimized to maximize the SE and EE without the need of inter-cell interference (ICI) management [5,6]. In practice, the network traffic fluctuates in different times and locations due to user behavior and mobility. This naturally calls for BS sleeping in the cells with low or no traffic load [7]. Yet it is unclear whether full frequency reuse is still optimal and whether optimal BS density still exists in the dense network when BS sleeping is allowed.

On the other hand, complicated ICI coordination is undesirable in UDNs with limited backhaul links. If we simply use traditional frequency planning, say dividing the entire frequency band into different subbands and assigning only one subband to a cell such that adjacent cells use orthogonal frequency subbands (e.g., with reuse factor 3), then what happens to the SE and EE of the networks with BS sleeping?

In this paper, we strive to quantify the SE and EE gain of full frequency reuse in UDNs, where the number of BSs may exceed the number of MSs and the BSs with no traffic load can be turned into sleep mode. To this end, we establish a unified framework for analyzing different frequency reuse strategies, including orthogonal, overlapped, and full frequency reuse among adjacent cells. We consider hexagonal cellular cells. On one hand, hexagonal topology is usually applied in the network planning and is mathematically tractable for the problem at hand. On the other hand, the MSs at the edge of each hexagonal cell experience most severe interference [8]. We derive closed form expressions of the SE and EE, and find the optimal strategies aiming to maximize SE and EE. Analytical results show that full frequency reuse is always SE-optimal and is EE-optimal only for UDNs, and the normalized gains increase with the BS density.

II. SYSTEM AND POWER CONSUMPTION MODEL

A. System Model

Consider an ultra dense wideband multi-input multi-output (MIMO) network, where multiple hexagonal cells are deployed. To facilitate analysis, we consider a hexagonal region of radius \( R \) with \( N_B \) small cells within the network, serving \( M \) users randomly distributed in the region without coordination. Each BS is equipped with \( N_t \) antennas, and each user is with a single antenna. To save energy and reduce ICI, a BS will be turned into sleep mode if no user is associated with the BS.
Without loss of generality and for notational simplicity, we assume that the frequency reuse factor \( L \) is in between one and three. As shown in Fig. 1, the overall frequency band of the network \( F \) is partitioned into three overlapped subbands with identical bandwidth \( f \), which are respectively used by adjacent three cells. To analyze the performance achieved by different frequency reuse strategies, we introduce an overlapping factor \( \omega \in [0, 1] \) to control the bandwidth of each subband as
\[
f = \frac{(1 + 2\omega)F}{3} \tag{1}
\]
When \( \omega = 0 \) or 1, \( f = F/3 \) or \( F \), which represents the network with \( L = 3 \) (i.e., orthogonal frequency reuse) or \( L = 1 \) (i.e., full frequency reuse), as shown in Fig. 1(b) or (d). When \( 0 < \omega < 1 \), the parts in the 2\(^{nd}\) and 3\(^{rd}\) subbands that are out of the frequency band of the network are shifted to the higher or lower frequency, as shown in Fig. 1(c).

Each BS serves multiple users in one of the subbands with zero forcing (ZF) precoding. Then, when \( 0 < \omega < 1 \), i.e., the three subbands overlap, each user may experience different classes of ICI in different parts of the subband, as shown in Fig. 2. We take a user using 1\(^{st}\) subband as an example to define four classes of ICI as follows. 1) The ICI is generated from other BSs using the 1\(^{st}\) subband. The set of these BSs is denoted as \( \Psi_1 \). 2) The ICI is from other BSs using the 1\(^{st}\) and 2\(^{nd}\) subbands. The set of these BSs is denoted as \( \Psi_2 \). 3) The ICI is from other BSs(81,911),(790,948) using the 1\(^{st}\) and 3\(^{rd}\) subbands, whose set is denoted as \( \Psi_3 \). These three classes of ICI are shown in Fig. 2(a). 4) The ICI is from all subbands, whose BS set is \( \Psi_4 \). This class of ICI is shown in Fig. 2(b).

For mathematical tractability, we fix the center frequency of each subband when \( 0 < \omega < 1 \), and set the center frequencies with separation of \( F/3 \). Then, each class of ICI for the users served in different subbands have identical aggregated bandwidth of \( f_iF \), where \( F \sum_{i=1}^{4} f_i = f \).

The signal to interference and noise ratio (SINR) of MS\(_m\) located in the \( l \)th cell who experiences the \( i \)th class of ICI can be expressed as
\[
SINR_{m,i} = \frac{P_l d_{ml}^{-\alpha} g_{ml}}{M_l (P_l \sum_{b \in \Psi_i} g_{mb} g_{mb} \xi_b + \sigma^2)} \tag{2}
\]
where \( P_l \) is the transmit power of each BS, \( d_{mb} \) is the distance between BS\(_b\) and MS\(_m\), \( \alpha \) is the path-loss exponent, \( g_{ml} \) is the equivalent channel gain from BS\(_b\) to MS\(_m\), including precoding gain and Rayleigh fading channel gain, which follows the Gamma distribution as \( g_{ml} \sim \Gamma(N_t - M_l + 1, 1) \) and \( g_{mb} \sim \Gamma(1, 1) \) [9]. \( M_l \) is the number of MSs in the \( l \)th cell with \( \sum_{b=1}^{N_m} M_l = \xi_b \) if BS\(_b\) is active and \( \xi_b = 0 \) if BS\(_b\) sleeps, and \( \sigma^2 \) is the variance of noise.

By using Shannon capacity formula, the SE of the \( N_B \)-cell region can be expressed as
\[
SE = \sum_{i=1}^{4} f_i \mathbb{E} \{ \log_2 (1 + SINR_{m,i}) \} \triangleq \sum_{i=1}^{4} \eta_i f_i \tag{3}
\]
where \( \eta_i = \mathbb{E} \{ \log_2 (1 + SINR_{m,i}) \} \), and the expectation is taken over small scale channel fading and user locations.

### B. Power Consumption Model

Because we study the impact of frequency reuse factor on EE, the relation between power consumption and bandwidth needs to be modelled. In EARTH Project [10], the power consumed by each module at the BS and its connection with bandwidth are characterized. Specifically, the powers consumed for baseband and a part of radio frequency (RF) transceiver, e.g. analogue-digital conversion (ADC), increase linearly with the bandwidth, while other parts do not depend on the bandwidth. Therefore, the power consumption of an active BS is a linear function of the bandwidth, which can be modelled as
\[
P_A = \frac{N_l f(P_{RF} + P_{BB}) + N_l P_{RF} + P_{t}/\rho PA}{(1 - \sigma_{DC})(1 - \sigma_{MS})} \tag{4}
\]
where \( P_{RF} \), \( P_{RF} \) and \( P_{BB} \) are respectively the powers consumed by ADC, the other parts of RF transceiver and baseband processor, \( \rho PA \) is the efficiency of the power amplifier, \( \sigma_{DC} \) and \( \sigma_{MS} \) are the loss factors of direct-current to direct-current power supply and main supply, respectively [11].

Upon substituting (1), the power consumption of an active BS can be expressed as
\[
P_A = \frac{N_l (1 + 2\omega) F(P_{RF} + P_{BB}) + 3N_l P_{RF} + 3P_{t}/\rho PA}{3(1 - \sigma_{DC})(1 - \sigma_{MS})} \tag{5}
\]
\[
= \frac{2N_l F(P_{RF} + P_{BB})}{3(1 - \sigma_{DC})(1 - \sigma_{MS})} \cdot \omega + \frac{N_l F(P_{RF} + P_{BB}) + 3N_l P_{RF} + 3P_{t}/\rho PA}{3(1 - \sigma_{DC})(1 - \sigma_{MS})} \triangleq k_A \cdot \omega + c_A
\]

The power consumption of a sleeping BS is a constant not associated with bandwidth [12], denoted as \( P_S \), which depends on the specific mode of BS sleeping. As defined in [10], in a light-sleep mode, only the power amplifier is switched off (also known as discontinuous transmission). In a deep-sleep mode, only the backhaul links among the BSs remains working.
Denote the power consumption of BSs as \( P_b \). Then, \( P_b = P_A \) when the BS is active, and \( P_b = P_S \) when it is sleeping. The average total power consumed by the BSs in the \( N_B \)-cell region is

\[
P_{\text{tot}} = E \left( \sum_{b=1}^{N_B} P_b \right) = N_B (p_A P_A + (1 - p_A) P_S) \tag{6}
\]

where \( p_A \) is the ratio of the number of active BSs to the total number of BSs. For the \( N_B \) hexagonal cells, this probability can be derived as

\[
p_A = 1 - \left( 1 - \frac{1}{N_B} \right)^M \tag{7}
\]

With (5) and (7), we can rewrite (6) as

\[
P_{\text{tot}} = p_A N_B (k_\omega \cdot \omega + c_A) + (1 - p_A) N_B P_S = p_A N_B k_\omega \cdot \omega + p_A N_B c_A + (1 - p_A) N_B P_S \triangleq k_{pc} \cdot \omega + c_{pc} \tag{8}
\]

III. FREQUENCY REUSE STRATEGY AND SE/EE ANALYSIS

In this section, we optimize the frequency reuse strategies to maximize the SE and EE of the \( N_B \)-cell region. Then, we show the potential of full frequency reuse in UDNs.

A. Optimal Frequency Reuse Strategies

The “bps/Hz/Watt” metric of EE is defined as [10]

\[
EE = \frac{SE}{P_{\text{tot}}} \tag{9}
\]

We optimize the reuse strategy by finding the optimal overlapping factor that maximizes the SE or EE. To this end, we first derive the SE as functions of \( \omega \). In the sequel, we express SE, EE and \( f_i \) in the form of functions of \( \omega \).

When \( \omega \leq 1/2 \), considering that the separation between the center frequencies of the subbands is \( F/3 \), we can obtain

\[
f_1(\omega) = -\frac{2}{3} \omega + \frac{1}{3} \tag{10a}
\]
\[
f_2(\omega) = f_3(\omega) = \frac{2}{3} \omega \tag{10b}
\]
\[
f_4(\omega) = 0 \tag{10c}
\]

which are linear functions of \( \omega \).

From (3) we can see that the SE is a linear function of \( f_i \). Therefore, the SE for the case of \( \omega \leq 1/2 \) is a linear function of \( \omega \), which can be derived by using Lagrange interpolation formula as

\[
SE(\omega)|_{\omega \leq 1/2} = 2 \left( SE \left( \frac{1}{2} \right) - SE \left( 0 \right) \right) \omega + SE(0) \triangleq k_1 \omega + c_1 \tag{11}
\]

Substituting (3) and (10) into (11), we obtain

\[
k_1 = 2M \sum_{i=1}^{4} \eta_i \left( f_i \left( \frac{1}{2} \right) - f_i \left( 0 \right) \right) = 2M \left( \eta_2 + \eta_3 - \eta_1 \right) \tag{12a}
\]
\[
c_1 = \frac{M}{3} \eta_1 \tag{12b}
\]

where \( \eta_i \) is defined in (3). To obtain a closed form expression of these two constants, we introduce an approximation of \( \eta_i \).

Prop. 1: The value of \( \eta_i \) can be approximated as

\[
\eta_i \approx \Phi_0 - \Phi_i + \log_2 \frac{N_t}{p_A} \tag{13}
\]

where \( \Phi_0 = \mathbb{E} \{ \log_2 d_{\text{num}}^{-\alpha} \}, \Phi_i = \mathbb{E} \{ \log_2 \sum_{b \in \Psi_i} d_{mb}^{-\alpha} \} \), both expectations are taken over the user location.

Proof: See Appendix A.

After substituting (13), the two constants in (12) can be approximated as

\[
k_1 \approx \frac{2M}{3} \left( \Phi_0 + \Phi_1 - 2 \Phi_2 + 2 \log_2 \frac{N_t}{p_A} \right) \tag{14a}
\]
\[
c_1 \approx \frac{M}{3} \left( \Phi_0 - \Phi_1 + 2 \log_2 \frac{N_t}{p_A} \right) \tag{14b}
\]

When \( \omega > 1/2 \), from (1) and the definitions of \( f_i \), we can derive that,

\[
f_1(\omega) = 0 \tag{15a}
\]
\[
f_2(\omega) = f_3(\omega) = -\frac{2}{3} \omega + \frac{2}{3} \tag{15b}
\]
\[
f_4(\omega) = 2 \omega - 1 \tag{15c}
\]

and the SE can be similarly derived as

\[
SE(\omega)|_{\omega > 1/2} = 2 \left( SE \left( 1 \right) - SE \left( \frac{1}{2} \right) \right) \omega - SE \left( 1 \right) + 2SE \left( \frac{1}{2} \right) \triangleq k_2 \omega + c_2 \tag{16}
\]

which is also a linear function of \( \omega \), where the two constants can be similarly approximated as

\[
k_2 \approx \frac{2M}{3} \left( \Phi_0 + 2 \Phi_2 - 3 \Phi_4 + 2 \log_2 \frac{N_t}{p_A} \right) \tag{17a}
\]
\[
c_2 \approx \frac{M}{3} \left( \Phi_0 - 4 \Phi_2 + 3 \Phi_4 + 2 \log_2 \frac{N_t}{p_A} \right) \tag{17b}
\]

The following proposition shows that the slopes of the two straight lines of \( SE(\omega)|_{\omega \leq 1/2} \) and \( SE(\omega)|_{\omega > 1/2} \) are approximately equal.

Prop. 2: For any given \( i, j = 0, \cdots, 4 \), \( \Phi_i - \Phi_j \) is a constant only depending on the path-loss exponent \( \alpha \). Moreover, the values of \( k_1 \) and \( k_2 \) are very close.

Proof: See Appendix B.

Proposition 2 suggests that the SE for all overlapping factors can be approximated as a linear function of \( \omega \) as

\[
SE(\omega) \approx (SE \left( 1 \right) - SE \left( 0 \right)) \omega + SE \left( 0 \right) \triangleq K \cdot \omega + C \tag{18}
\]

where the two constants can be approximated as

\[
K \approx \frac{M}{3} \left( 2 \Phi_0 + \Phi_1 - 3 \Phi_4 + 2 \log_2 \frac{N_t}{p_A} \right) \tag{19a}
\]
\[
C \approx \frac{M}{3} \left( \Phi_0 - \Phi_1 + 2 \log_2 \frac{N_t}{p_A} \right) \tag{19b}
\]

By taking the numerical integration for all practical values of \( \alpha \) (i.e., \( 2 \leq \alpha \leq 6 \)), we can show that the sum of the first
three terms of $K, 2\Phi_0 + \Phi_1 - 3\Phi_4$, is positive. Therefore, $K$ is always positive. This implies that the SE always achieve its maximum with full frequency reuse.

Substituting (8) and (18) into (9), we can express the EE as

$$EE(\omega) = \frac{K\omega + C}{k_{pc}\omega + c_{pc}}$$

When $k_{pc}/c_{pc} > K/C$, $EE(\omega)$ is a decreasing function of $\omega$. Then, the optimal overlapping factor $\omega^{opt} = 0$. Conversely, when $k_{pc}/c_{pc} < K/C$, $\omega^{opt} = 1$. This implies that the EE achieves its maximum with either orthogonal or full frequency reuse.

By defining a Heaviside step function $H(\cdot)$, where $H(x) = 0$ when $x < 0$ and $H(x) = 1$ otherwise, the optimal overlapping factor can be rewritten as

$$\omega^{opt} = H\left(\frac{K}{C} - \frac{k_{pc}}{c_{pc}}\right)$$

whose relation with the number of BSs is shown as follows.

**Prop. 3:** $\phi(N_B) \equiv \frac{K}{C} - \frac{k_{pc}}{c_{pc}}$ increases with $N_B$.

After substituting $k_{pc}, c_{pc}, K$ and $C$ defined in (8) and (19) into $\phi(N_B)$, the proposition is easy to prove by taking derivative of $\phi(N_B)$. Proposition 3 implies that the root of

$$\phi(N_B) = 0$$

denoted as $\Theta_B$, is a threshold for the number of BSs. When $N_B > \Theta_B$, $\omega^{opt} = 1$, otherwise $\omega^{opt} = 0$.

(22) is a transcendental equation whose solution can be obtained numerically. To gain useful insight, we introduce a polynomial approximation of $\phi(N_B)$, from which and (7) we can find a closed form solution of the threshold.

**Prop. 4:** The threshold can be approximated as

$$\Theta_B \approx \left(1 - (1 - p_A^{*})^{19}\right)^{-1} \approx \frac{M}{p_A^*}$$

where $p_A^*$ is the root of following quadratic equation of $p_A$

$$(k_A - 2c_A + 2P_S)p_A^2 + (-2P_S + (c_A - P_S)(K\ln 2 + 2) - k_A(C\ln 2 + 1))p_A + P_S(K\ln 2 + 2) = 0$$

and $K = 2\Phi_0 + \Phi_1 - 3\Phi_4 + 2\log_2 N_t$, $C = \Phi_0 - \Phi_1 + \log_2 N_t$.

From (7) we can see that $N_B$ is a function of $p_A$. Then (22) can be expressed as an equation of $p_A$. By substituting the Taylor series approximation $\log_2 p_A \approx (p_A - 1)/\ln 2$ into this equation, the proposition can be proved.

Proposition 4 implies that whether orthogonal or full frequency reuse is EE-optimal is approximately determined by the ratio of BS density and MS density, $N_B/M$.

**B. Gain of Full Frequency Reuse in UDNs**

From previous analysis we can see that when $N_B/M > p_A^{* - 1}$, $\omega^{opt} = 1$ (i.e., full frequency reuse is EE-optimal), otherwise $\omega^{opt} = 0$. We refer to the networks satisfying $N_B/M > p_A^{* - 1}$ as UDNs. In what follows, we derive the SE and EE gains of full frequency reuse over orthogonal frequency reuse.

1) **SE Gain:** From (18) and (19), it is not hard to derive the SE gain for the considered network with $L = 3$ as

$$SE(1) - SE(0) = K$$

which increases with $N_t$ and decrease with $p_A$. Because $p_A$ decreases with $N_B$ as shown in (7), we know from the Chain Rule that the SE gain increases with $N_B$.

When $N_B \rightarrow \infty$, the normalized SE gain is obtained as

$$\lim_{N_B \rightarrow \infty} \frac{SE(1) - SE(0)}{SE(0)} = \lim_{N_B \rightarrow \infty} 2 - \frac{2\Phi_0 + \Phi_1 - 3\Phi_4 + 2\log_2 N_t}{\Phi_0 - \Phi_1 + \log_2 N_t} = 2$$

which is an increasing function of $N_B$ and approaches 2.

In fact, such a SE gain in UDNs depends on the frequency reuse factor. Specifically, when the reuse factor is $L$, we can show that the normalized SE gain approaches $L - 1$.

2) **EE Gain:** From (8), (19), and (20), we can derive the limit of the EE gain for the network with $L = 3$ as

$$\lim_{N_B \rightarrow \infty} \frac{EE(1) - EE(0)}{EE(0)} = \lim_{N_B \rightarrow \infty} -\frac{2M \log_2 p_A}{3N_B P_S}$$

(23)

$$\lim_{N_B \rightarrow \infty} -\frac{2M}{3N_B P_S} \log_2 \frac{M}{N_B} = 0$$

where (a) is because $p_A = M/N_B - (M^2)/N_B^2 + \cdots = O(M/N_B)$ by using binomial expansion to (7). We can see that the EE gain deceases with $N_B$ when $N_t$ is large enough. This is because the SE gain increases logarithmically while the power consumption increases linearly with $N_B$. The limit of normalized EE gain can be similarly derived as

$$\lim_{N_B \rightarrow \infty} \frac{EE(1) - EE(0)}{EE(0)} = \lim_{N_B \rightarrow \infty} -\frac{2M \log_2 \frac{M}{N_B}}{3N_B P_S} = 2$$

(24)

which is also an increasing function of $N_B$ and approaches 2.

Again, we can show that when the frequency reuse factor is $L$, this gain approaches $L - 1$.

**IV. NUMERICAL AND SIMULATION RESULTS**

In this section, we validate previous analysis and compare the results for UDNs with traditional cellular networks via numerical and simulation results.

In the simulation, the radius of the $N_B$-cell hexagonal region is 250 m, which includes 1, 7, 19, 37, 91 or 1027 (one to nineteen tiers of) cells. Wrap around is considered to remove the edge effect, hence even when $N_B = 1$ there are multiple cells surrounded the considered reference cell. Except the network with $N_B = 1$ in the region, each BS is equipped with four antennas. 10 MSs are randomly distributed in the region of 250 m. The path loss exponent is 3.76, the average power loss at the reference distance of 1 m is 36.3 dB, the variance of noise is $-95$ dBm. The other simulation setup is based
on the parameters for a pico cell in [10], where the power consumption parameters in (4) are $P_{RF_1} = 0.4$ W, $P_{RF_2} = 0.4$ W, $P_{BB_1} = 1.5$ W, $\rho_{PA} = 8\%$, $\sigma_{DC} = 8\%$ and $\sigma_{MS} = 10\%$, respectively. The transmit power is $P_t = 0.13$ W. The power consumption of a sleeping BS is 2.4 W. All simulation results are obtained by averaging over 100 realizations of small scale fading channels and 100 random locations of the MSs.

To evaluate the accuracy of the approximation in Section III, we provide the simulation and numerical results of SE in Fig. 3, where the numerical results are obtained from (18) when $N_B = 37$ or 91. It is shown that the numerical and simulation results are close. Moreover, both $SE(\omega)_{|\omega \leq 1/2}$ and $SE(\omega)_{|\omega > 1/2}$ are linear functions of $\omega$ and the corresponding slopes are close. This validates Proposition 2.

In Fig. 3, we also compare the SE-$\omega$ relation of UDNs with traditional cellular networks. Since we are not concerned about SEs themselves of these two networks, a comparable parameter setting is unnecessary. In the traditional cellular network, one BS equipped with 16 antennas serves all the 10 users in the cell of 250 m with ZF percoding. We can see that in both kind of networks, the SE achieves its maximum with full frequency reuse, and the gain of full frequency reuse over orthogonal frequency reuse increases with the number of BSs. This agrees with our analysis in Section III.

In Fig. 4, we compare the simulated EE of UDNs and traditional cellular networks with two frequency reuse strategies. The value of $\Omega_B$ numerically obtained from (22) and the approximation obtained from Proposition 4 are 11 and 12, respectively, which indicates that the approximation is accurate. This also shows that for the considered system setup the BS-MS density ratio that bifurcates optimal EE design between full and orthogonal frequency reuse is $N_B/M \approx 1$. Because there are 10 users in the $N_B$-cell region, $N_B = 1$ or 7 can reflect the traditional network with $N_B/M$ approximately 1/10 or 1/1, and $N_B = 19$, 37, or 91 can reflect the UDN with $N_B/M$ approximately 2/1, 4/1 and 100/1. Moreover, we also provide the results for a scenario where $N_B = 1027$ (19 tiers of small cells in the 250 m region and $N_B/M = 100$) to show the performance limits of practical UDNs. The results are consistent with our analysis in Section III. Moreover, the normalized EE gain of full frequency reuse over orthogonal frequency reuse increases with $N_B$ slowly, which is far less than 200% even when $N_B = 1000$.

V. CONCLUSION

In this paper, we investigated the EE optimal frequency reuse strategies in cellular networks with BS sleeping. We confirm the intuition that full frequency reuse is always SE optimal. However, full frequency reuse is EE optimal only when the ratio of BS density to user density exceeds a threshold (i.e., in UDNs), where the threshold depends on the number of antennas at each BS, circuit power consumption parameters, and path loss exponent. When the ratio is less than the threshold, orthogonal frequency reuse is EE optimal. Both of the normalized SE and EE gains of full frequency reuse over the orthogonal frequency reuse in UDNs increase with the number of BSs and finally approach a constant, which equals to the frequency reuse factor minus one.

APPENDIX A

PROOF OF PROP. 1

When the network is dense, the SINR and ICI power of each user are much higher than 1 and noise variance, respectively, and almost no BS serves two or more MSs. Thus, the terms “1”, “$2^\infty$” and “$M_i$” inside the log function in $\eta_i$ defined in (3) can be ignored. Denote

$$\varphi_{m,i} \triangleq \sum_{b \in \Psi_i} d_{mb}^{-\alpha} g_{mb} \xi_b$$ \hspace{1cm} (A.1)

From (2) and (A.1), $\eta_i$ can be approximated as

$$\eta_i \approx \mathbb{E} \left\{ \log_2 \left( \frac{d_{mi}^{-\alpha} g_{mi}}{\varphi_{m,i}} \right) \right\}$$ \hspace{1cm} (A.2)

$$= \mathbb{E} \left\{ \log_2 \left( d_{mi}^{-\alpha} g_{mi} \right) \right\} - \mathbb{E} \{ \log_2 \varphi_{m,i} \}$$
The first term in (A.2) can be expressed as
\[
\mathbb{E} \{ \log_2 (d_{ml}^{-\alpha} g_{ml}) \} = \mathbb{E} \{ \log_2 d_{ml}^{-\alpha} \} + \mathbb{E} \{ \log_2 g_{ml} \}
\]
where \((a)\) is obtained by the nature of Gamma distribution because \(g_{ml} \sim \Gamma(N_i - M_i + 1, 1)\), \(\psi(n)\) is the Digamma function, which has a very accurate approximation as \(\psi(n) \approx \ln(n)\) when \(n\) is a positive integer. \(\Phi_0 = \mathbb{E}_d \{ \log_2 d_{ml}^{-\alpha} \},\) where the expectation is taken over the location of MSSs. This approximation is accurate when the number of BSs is large.

From (A.1), the second term in (A.2) can be expressed as
\[
\mathbb{E} \{ \log_2 p_{i,m,i} \} = \mathbb{E}_d \left\{ \mathbb{E}_g, \mathbb{E}_{\xi} \left( \log_2 \left( \sum_{b \in \Psi_i} d_{mb}^{-\alpha} g_{mb} \xi_b \right) \right) \right\}
\]
Because \(\mathbb{E} \{ \xi_i \} = P(\xi_i = 1) = p_A\), (A.4) can be derived as
\[
\mathbb{E} \{ \log_2 p_{i,m,i} \} \approx \mathbb{E}_d \left\{ \log_2 \left( \sum_{b \in \Psi_i} d_{mb}^{-\alpha} g_{mb} \xi_b \right) \right\}
\]
\[
= \mathbb{E}_d \left\{ \log_2 \left( \sum_{b \in \Psi_i} d_{mb}^{-\alpha} \right) + \log_2 p_A \right\}
\]
\[
\Delta = \Phi_i + \log_2 p_A
\]
where \((a)\) is because \(\mathbb{E} \{ \log_2 (x) \} \approx \log_2 (\mathbb{E} \{ x \})\), which is accurate with a small number of BSs, \(\mathbb{E}_g \{ g_{mb} \} = 1\) because \(g_{mb} \sim \Gamma(1, 1)\), and \(\Phi_i = \mathbb{E}_d \{ \log_2 \sum_{b \in \Psi_i} d_{mb}^{-\alpha} \}.\) From the definition of four classes of ICI, we know that \(\Phi_1 < \Phi_2 = \Phi_3 < \Phi_4.\)

Substituting (A.3) and (A.5) into (A.2), we have
\[
\eta_i \approx \Phi_0 - \Phi_i + \log_2 \frac{N_i}{p_A}
\]

**APPENDIX B**

**PROOF OF PROP. 2**

In the considered network, the radius of each small cell is \(r = R / \sqrt{N_B}\). To denote the location of each BS, we build a Cartesian coordinate system, where the location of BS\(_m\) is the origin of coordinates. Then, the coordinate of BS\(_m\) can be denoted as \((x_m, y_m)\). Because the distance between any BS and BS\(_m\) is proportional to \(r\), \(x_m\) and \(y_m\) are also proportional to \(r\), i.e., \(x_m/r\) and \(y_m/r\) do not depend on \(r\). From the definition of \(\Phi_0\) after (A.3), we have
\[
\Phi_0 = \int_T \int_{S_T} \frac{1}{S_T} \log_2 \left( x^2 + y^2 \right)^{-\frac{\alpha}{2}} \, dx \, dy
\]
\[
= \int_T \int_{S_T} \frac{1}{S_T} \log_2 \left( r^2u^2 + r^2v^2 \right)^{-\frac{\alpha}{2}} r^2 u^2 \, du \, dv
\]
\[
= \int_T \int_{S_T} \frac{1}{S_T} \log_2 \left( u^2 + v^2 \right)^{-\frac{\alpha}{2}} + \log_2 r^{-\alpha} \, du \, dv
\]
\[
= \int_T \int_{S_T} \log_2 \left( u^2 + v^2 \right)^{-\frac{\alpha}{2}} du \, dv + \log_2 r^{-\alpha}
\]
where \(T\) and \(G\) are hexagonal regions with radius \(r\) and unit radius, respectively, \(S_T\) and \(S_G\) are the areas of \(T\) and \(G\), \(u = x/r\), and \(v = y/r\). Because \(G\) and \(S_G\) do not depend on \(r\), the first term \(\int_{S_G} \frac{1}{S_G} \log_2 \left( u^2 + v^2 \right)^{-\frac{\alpha}{2}} \, du \, dv\) in (B.1) only depends on the path-loss exponent \(\alpha\).

With similar way in deriving (B.1), we can rewrite \(\Phi_i\) as
\[
\Phi_i = \int_T \int_{S_T} \frac{1}{S_T} \log_2 \sum_{b \in \Psi_i} \left( \frac{u - x_b}{r} \right)^2 + \left( \frac{v - y_b}{r} \right)^2 \right)^{-\frac{\alpha}{2}} \, du \, dv
\]
\[
+ \log_2 r^{-\alpha}
\]
(B.2)

Therefore, we have
\[
\Phi_0 - \Phi_i = \int_T \int_{S_T} \log_2 \sum_{b \in \Psi_i} \left( \frac{u - x_b}{r} \right)^2 + \left( \frac{v - y_b}{r} \right)^2 \right)^{-\frac{\alpha}{2}} \, du \, dv
\]
which only depends on \(\alpha\).

Because \(\Phi_i - \Phi_j = (\Phi_j - \Phi_0) - (\Phi_i - \Phi_0)\), \(\Phi_i - \Phi_j\) also depends on \(\alpha\) for any given \(i, j\).

Moreover, from (14) and (17), we have
\[
\frac{k_1}{k_2} = \Phi_0 + \Phi_i - 2\Phi_2 + \log_2 \frac{N_i}{p_A}
\]
\[
= \Phi_0 + 2\Phi_2 - 3\Phi_4 + \log_2 \frac{N_i}{p_A}
\]
\[
= 1 + \Phi_0 + 4\Phi_2 - 3\Phi_4 + \log_2 \frac{N_i}{p_A}
\]
(B.3)

By taking the numerical integration for all practical values of \(\alpha\) (i.e., \(2 \leq \alpha \leq 6\)), we can show that \(\Phi_1 - 4\Phi_2 + 3\Phi_4 \ll \Phi_0 + 2\Phi_2 - 3\Phi_4 + \log_2 \frac{N_i}{p_A}\) Thus we have \(k_1 \approx k_2\).

**REFERENCES**


