A Unified Analysis of Spectral Efficiency for Two-Hop Relay Systems With Different Resource Configurations

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***Abstract*—In this paper, we analyze in a unified way the achievable rate and spectral efficiency of multicarrier and multiantenna two-hop relay systems with an interference-free linear transceiver. We consider two kinds of decode-and-forward (DF) relay systems with three nodes. In the first kind of relay systems where only the relay node employs the full channel state information (CSI) of both the source–relay (S–R) link (first hop) and the relay–destination (R–D) link (second hop), the relay eliminates the interference of these links. In the second kind of systems, where the source node employs the CSI of the first hop, the destination node employs the CSI of the second hop, whereas the relay node employs the full CSI of both the hops; these nodes can jointly eliminate the interference. For the multicarrier relay or multiantenna relay systems supporting different diversity and multiplexing gains and under the aforementioned assumptions for the CSI, we find that the achievable rate of each hop or its lower bound can be unified in the generalized mean of the eigenvalues of an equivalent channel correlation matrix. Furthermore, by using the properties of generalized mean and by resorting to the random matrix theory, we derive the asymptotic spectral efficiency of the multicarrier or multiantenna systems with different CSI assumptions. Finally, we provide simulation results to validate our analytical results. Our studies show that the multicarrier systems using code-division multiplexing (CDM) or the multiantenna systems can benefit more from exploiting the CSI at all the three nodes than the multicarrier systems using frequency-division multiplexing (FDM), particularly, in the case of a high load factor.**

***Index Terms*—Diversity, generalized mean, multiantenna, multicarrier, multiplexing, spectral efficiency, two-hop relay, unified analysis.**

I. Introduction

RELAY transmission is able to enhance reliability and increase coverage and capacity of wireless systems [1]–[3], which has attracted significant research interests in decades. In a relay system, the channel state information (CSI) available to its nodes has a critical impact on the achievable performance. To acquire the CSI at each node, training symbols are required for channel estimation, and channel reciprocity needs to be exploited in time-division duplex (TDD) systems. In a three-node two-hop relay system, to assist the relay to estimate the channel coefficients of both the source–relay (S–R) link and the relay–destination (R–D) link, the source and destination nodes need to send mutually orthogonal training symbols. Then, the relay can estimate all the channels from the two nodes to it and further obtain the CSI from itself to the destination by using the channel reciprocity if TDD is assumed. To assist the source and the destination to estimate the channel coefficients in the S–R link and the R–D link, respectively, the relay needs to broadcast one training symbol to the two nodes. Then, the source can obtain its channel to the relay again using the channel reciprocity. In frequency-division duplex systems, extra feedback is required in the S–R link and the R–D link to provide all CSI to the three nodes. It is undoubted that sharing the CSI among the source, relay, and destination nodes can provide high spectral efficiency by the joint design of the transceivers for the three nodes. However, this will inevitably lead to a large training or feedback overhead [4, 5]. In fact, in addition to the channel knowledge, the performance of relay systems also depends on various system settings and transmission strategies, e.g., system load factor, frequency or spatial resource employed for conveying signals, a multiplexing method used to separate multiple data streams, etc. With the aid of a unified analysis of the spectral efficiency of the relay systems with different CSI and different settings, we may shed light upon the following question: In what kind of relay systems is exploiting CSI by all nodes beneficial to improving spectral efficiency?

Multicarrier and multiple-input–multiple-output (MIMO) systems are two key techniques in many current development and research efforts toward high-throughput wireless communications. There are many similarities in modeling and analyzing multicarrier systems and multiantenna systems. Nonetheless, the two kinds of systems are distinctive in many aspects, e.g., channel features. Specifically, the channel matrices of multicarrier systems are diagonal, but in general, the channel matrices of MIMO systems do not have any particular structure. The difference in channel feature leads to the difference in spectral efficiency of the two kinds of systems. In [6], the authors provided a unified performance expression of multicarrier systems when transmitting only one data stream under the full-diversity

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scenario. The results are only applicable to analyze the spectral efficiency with a low load factor. In [7], the authors presented a unified bit-error-rate (BER) analysis for group-orthogonal code-division multiplexing (CDM) multicarrier MIMO systems, where only the receiver uses CSI. These unified results cannot be extended to analyze the spectral efficiency of the multicarrier or MIMO relay system transmitting multiple data streams.

In this paper, we investigate multicarrier and MIMO two-hop relay systems, where a source node transmits information to a destination node assisted by a decode-and-forward (DF) relay node. When the interference-free linear transceiver is considered for the two kinds of systems, we have an interesting finding: when operated in a high-SNR region, the achievable rates under different assumptions for exploiting CSI can be expressed as the generalized mean of different orders. Under this observation, then, by exploiting the properties of the generalized mean [8] and by applying the random matrix theory in the asymptotic region [9], we derive the closed-form expressions of the asymptotic spectral efficiency for both the multicarrier and MIMO two-hop relay systems. Simulation and numerical results validate our analytical analysis and demonstrate the impact of the available channel knowledge on the performance of the considered relay systems.

The remainder of this paper is organized as follows. In Section II, we describe the signal model and transmission scheme. The main results of the unified analysis and the corresponding proofs are presented in Section III and IV. Simulation and numerical results are provided in Section V, and finally, our conclusions are drawn in Section VI.

Notations: Conjugation, transpose, Hermitian transpose, and expectation are represented by $(\cdot)^*$, $(\cdot)^T$, $(\cdot)^H$, and $\mathbb{E}\{\cdot\}$, respectively. The trace of a square matrix is denoted by $\mathrm{tr}\{\cdot\}$, and $\mathrm{diag}\{\cdot\}$ denotes the diagonal matrix.

II. System Description

We consider a multicarrier or MIMO two-hop relay system, where half-duplex DF relay $\mathcal{R}$ assists source $\mathcal{S}$ to transmit K data streams to destination $\mathcal{D}$. The source, the relay, and the destination have $M_\mathcal{S}$, $M_\mathcal{R}$, and $M_\mathcal{D}$ antennas or orthogonal subcarriers, respectively. In the multicarrier system, $M_\mathcal{S} = M_\mathcal{R} = M_\mathcal{D} = M$. In the MIMO system, $M_\mathcal{S} = M_\mathcal{D} = K$, whereas $M_\mathcal{R} = M$. The load factor is $\beta = K/M$.

In the first hop, source $\mathcal{S}$ transmits data streams to relay $\mathcal{R}$. The estimated symbol vector at the relay can be expressed as

$$\hat{\boldsymbol{d}}_\mathcal{R} = \boldsymbol{W}_{\mathcal{R}r}^H \boldsymbol{H}_\mathcal{S} \boldsymbol{W}_\mathcal{S} \boldsymbol{d} + \boldsymbol{W}_{\mathcal{R}r}^H \boldsymbol{n}_\mathcal{R} \tag{1}$$

where $\boldsymbol{d} = [d_1, \ldots, d_K]^T$ represents the transmit symbol vector, and all its entries $\{d_k, k = 1, \ldots, K\}$ are independent and identically distributed (i.i.d.) random variables satisfying $\mathbb{E}\{d_k\} = 0$ and $\mathbb{E}\{|d_k|^2\} = P_\mathcal{S}$, with $P_\mathcal{S}$ expressing the maximal transmit power per symbol at the source node; $\boldsymbol{W}_\mathcal{S} = [\boldsymbol{w}_{\mathcal{S}_1}, \ldots, \boldsymbol{w}_{\mathcal{S}_K}] \in \mathbb{C}^{M_\mathcal{S} \times K}$ denotes the precoding matrix at the source satisfying $\boldsymbol{w}_{\mathcal{S}_k}^H \boldsymbol{w}_{\mathcal{S}_k} = 1$ to meet the transmit power constraint; $\boldsymbol{H}_\mathcal{S} \in \mathbb{C}^{M_\mathcal{R} \times M_\mathcal{S}}$ is the channel matrix in the S–R link; $\boldsymbol{W}_{\mathcal{R}r} = [\boldsymbol{w}_{\mathcal{R}r_1}, \ldots, \boldsymbol{w}_{\mathcal{R}r_K}] \in \mathbb{C}^{M_\mathcal{S} \times K}$ denotes the receiving matrix at the relay; and $\boldsymbol{n}_\mathcal{R}$ is the zero-mean Gaussian noise vector at the relay with a covariance matrix $\mathbb{E}\{\boldsymbol{n}_\mathcal{R} \boldsymbol{n}_\mathcal{R}^H\} = \sigma_\mathcal{R}^2 \boldsymbol{I}_{M_\mathcal{R}}$, with $\sigma_\mathcal{R}^2$ being the noise variance.

In the second hop, relay $\mathcal{R}$ first decodes symbol vector $\boldsymbol{d}_\mathcal{R}$ from $\hat{\boldsymbol{d}}_\mathcal{R}$ and then forwards $\boldsymbol{d}_\mathcal{R}$ to destination $\mathcal{D}$. The estimated symbol at the destination can be expressed as

$$\hat{d}_\mathcal{D} = \boldsymbol{W}_\mathcal{D}^H \boldsymbol{H}_\mathcal{D} \boldsymbol{W}_{\mathcal{R}t} \boldsymbol{d}_\mathcal{R} + \boldsymbol{W}_\mathcal{D}^H \boldsymbol{n}_\mathcal{D} \tag{2}$$

where $\boldsymbol{d}_\mathcal{R} = [d_{\mathcal{R}_1}, \ldots, d_{\mathcal{R}_K}]^T$, satisfying $\mathbb{E}\{d_{\mathcal{R}_k}\} = 0$ and $\mathbb{E}\{|d_{\mathcal{R}_k}|^2\} = P_\mathcal{R}$, with $P_\mathcal{R}$ denoting the maximal transmit power per symbol at the relay node; $\boldsymbol{W}_{\mathcal{R}t} = [\boldsymbol{w}_{\mathcal{R}t_1}, \ldots, \boldsymbol{w}_{\mathcal{R}t_K}] \in \mathbb{C}^{M_\mathcal{S} \times K}$ denotes the precoding matrix at the relay satisfying $\boldsymbol{w}_{\mathcal{R}t_k}^H \boldsymbol{w}_{\mathcal{R}t_k} = 1$; $\boldsymbol{H}_\mathcal{D} \in \mathbb{C}^{M_\mathcal{D} \times M_\mathcal{R}}$ is the channel matrix for the R–D link; $\boldsymbol{W}_\mathcal{D} = [\boldsymbol{w}_{\mathcal{D}_1}, \ldots, \boldsymbol{w}_{\mathcal{D}_K}] \in \mathbb{C}^{M_\mathcal{S} \times K}$ denotes the receiving matrix at the destination; $\boldsymbol{n}_\mathcal{D}$ is the zero-mean Gaussian noise vector at the destination with covariance matrix $\mathbb{E}\{\boldsymbol{n}_\mathcal{D} \boldsymbol{n}_\mathcal{D}^H\} = \sigma_\mathcal{D}^2 \boldsymbol{I}_{M_\mathcal{D}}$, where $\sigma_\mathcal{D}^2$ is the noise variance.

To achieve interference-free transmission at each hop by using linear processing, the number of data streams should satisfy $1 \leq K \leq M$ (i.e., $0 < \beta \leq 1$). Then, the precoder at source $\mathcal{S}$, the transceiver at relay $\mathcal{R}$, and the receiver at destination $\mathcal{D}$ satisfy

$$\boldsymbol{w}_{\mathcal{R}r_j}^H \boldsymbol{H}_\mathcal{S} \boldsymbol{w}_{\mathcal{S}_i} = 0 \quad \boldsymbol{w}_{\mathcal{D}_i}^H \boldsymbol{H}_\mathcal{D} \boldsymbol{w}_{\mathcal{R}t_j} = 0 \qquad \forall i \neq j. \tag{3}$$

In this case, the SNRs of the S–R and R–D links with respect to the kth symbol are

$$\gamma_{\mathcal{S}_k} = \frac{P_\mathcal{S}}{\sigma_\mathcal{R}^2} \frac{|\boldsymbol{w}_{\mathcal{R}r_k}^H \boldsymbol{H}_\mathcal{S} \boldsymbol{w}_{\mathcal{S}_k}|^2}{\boldsymbol{w}_{\mathcal{R}r_k}^H \boldsymbol{w}_{\mathcal{R}r_k}} \tag{4a}$$

$$\gamma_{\mathcal{D}_k} = \frac{P_\mathcal{R}}{\sigma_\mathcal{D}^2} \frac{|\boldsymbol{w}_{\mathcal{D}_k}^H \boldsymbol{H}_\mathcal{D} \boldsymbol{w}_{\mathcal{R}t_k}|^2}{\boldsymbol{w}_{\mathcal{D}_k}^H \boldsymbol{w}_{\mathcal{D}_k}}. \tag{4b}$$

The achievable rates in the S–R and R–D links are $KR_\mathcal{S}$ and $KR_\mathcal{D}$, respectively, where

$$R_\mathcal{S} \triangleq \frac{1}{K} \sum_{k=1}^{K} \log\left(1 + \gamma_{\mathcal{S}_k}\right) \tag{5a}$$

$$R_\mathcal{D} \triangleq \frac{1}{K} \sum_{k=1}^{K} \log\left(1 + \gamma_{\mathcal{D}_k}\right) \tag{5b}$$

denote the normalized achievable rate over the number of data streams. We refer to them as the rate per data stream.

Considering that, in the forthcoming analysis, the asymptotic achievable rates will be infinite as $K \to \infty$, whereas the asymptotic rate per data stream always keeps finite and is easy to analyze, we mainly investigate the rate per data stream.

The attainable rate of the two-hop DF relay system can be expressed as KR, where $R = \min\{\alpha R_\mathcal{S}, (1-\alpha) R_\mathcal{D}\}$ is the end-to-end achievable rate per data stream, where α and $(1 - \alpha)$ denote the portions of time that the first and second hops transmit, respectively. Explicitly, the attainable rate is obtained based on the time-division principles. Furthermore, it is not difficult to understand that the optimal time allocation maximizing the achievable rate makes both the hops have the same rate, i.e.,

TABLE I
TYPICAL GENERALIZED MEANS

$\mathcal{M}_p$	$\mathcal{M}_p(x_1, \cdots, x_M)$	Mean
$\mathcal{M}_\infty$	$\lim_{p\to\infty} \mathcal{M}_p(x_1, \cdots, x_M)$ $= \max\{x_1, \cdots, x_M\}$	Maximum
$\mathcal{M}_2$	$\sqrt{\sum_{m=1}^M x_m^2/M}$	Root mean square
$\mathcal{M}_1$	$\sum_{m=1}^M x_m/M$	Arithmetic mean
$\mathcal{M}_0$	$\lim_{p\to 0} \mathcal{M}_p(x_1, \cdots, x_M)$ $= \sqrt[M]{\prod_{m=1}^M x_m}$ $= \exp\left\{\sum_{m=1}^M \ln x_m/M\right\}$	Geometric mean
$\mathcal{M}_{-\frac{1}{2}}$	$\left(\sum_{m=1}^M x_m^{-\frac{1}{2}}/M\right)^{-2}$	Square harmonic mean root
$\mathcal{M}_{-1}$	$\left(\sum_{m=1}^M x_m^{-1}/M\right)^{-1}$	Harmonic mean
$\mathcal{M}_{-\infty}$	$\lim_{p\to -\infty} \mathcal{M}_p(x_1, \cdots, x_M)$ $= \min\{x_1, \cdots, x_M\}$	Minimum

$\alpha R_\mathcal{S} = (1-\alpha)R_\mathcal{D}$, which yields $\alpha = R_\mathcal{D}/(R_\mathcal{S} + R_\mathcal{D})$. In this case, we have

$$R = \frac{1}{\frac{1}{R_\mathcal{S}} + \frac{1}{R_\mathcal{D}}} = \frac{R_\mathcal{S} R_\mathcal{D}}{R_\mathcal{S} + R_\mathcal{D}} \tag{6}$$

and the average spectral efficiency is

$$\eta = \beta \mathbb{E}\{R\}. \tag{7}$$

III. Main Results of the Unified Analysis

Here, we present the main results obtained from the unified analysis and leave the proof to the succeeding section. We first employ the generalized mean to unify the expressions for the achievable rates of both the multicarrier and MIMO relay systems, when the two kinds of assumptions for exploiting CSI are considered. We then employ random matrix theory and exploit the properties of the generalized mean to derive the asymptotic spectral efficiency values of these systems.

For readers' convenience, we first briefly introduce the definition and properties of a generalized mean.

Definition 1—Generalized Mean: Let $x_1, \ldots, x_M$ be positive real numbers and p be a real number. The generalized mean of $x_1, \ldots, x_M$ with exponent p is defined as

$$\mathcal{M}_p(x_1, \ldots, x_M) \triangleq \left(\frac{1}{M}\sum_{m=1}^M x_m^p\right)^{\frac{1}{p}} \tag{8}$$

where p is called the *order of mean*.

The generalized mean is also known as the *power mean* or the *Hölder mean*. In Table I, we list some special means obtained from the generalized mean associated with the corresponding values for parameter p [8].

The generalized mean has the following properties.

Property 1: If $x_m \geq y_m > 0$ for $m = 1, \ldots, M$, we have

$$\mathcal{M}_p(x_1, \ldots, x_M) \geq \mathcal{M}_p(y_1, \ldots, y_M). \tag{9}$$

Property 2: For arbitrary p, q, if $p > q$, we have

$$\mathcal{M}_p(x_1, \ldots, x_M) \geq \mathcal{M}_q(x_1, \ldots, x_M) \tag{10}$$

and the equality holds if and only if $x_1 = \cdots = x_M$.

Property 3:

$$\begin{aligned}\log\left(1 + \mathcal{M}_1(x_1, \ldots, x_M)\right) &\geq \frac{1}{M}\sum_{m=1}^M \log(1 + x_m) \\ &\geq \log\left(1 + \mathcal{M}_0(x_1, \ldots, x_M)\right)\end{aligned} \tag{11}$$

where the equality holds if and only if $x_1 = \cdots = x_M$.

Note that, Properties 1 and 2 are from [8]. The proof of Property 3 is provided in the Appendix.

Below, we investigate the performance of the relay systems, when the following two typical assumptions for using CSI are respectively made.

- **CSI-R**: The CSI of both the hops is only available at the relay, i.e., only relay node $\mathcal{R}$ has perfect knowledge of $\boldsymbol{H}_\mathcal{S}$ and $\boldsymbol{H}_\mathcal{D}$, whereas the source and destination nodes $\mathcal{S}$ and $\mathcal{D}$, respectively, do not have any CSI.
- **CSI-SRD**: Source node $\mathcal{S}$ has perfect knowledge of $\boldsymbol{H}_\mathcal{S}$, destination node $\mathcal{D}$ has perfect knowledge of $\boldsymbol{H}_\mathcal{D}$, and relay node $\mathcal{R}$ has perfect knowledge of both $\boldsymbol{H}_\mathcal{S}$ and $\boldsymbol{H}_\mathcal{D}$.

We will analyze the transmission strategies for different numbers of data streams as follows.

- **Full diversity**: The extreme case of $K = 1$, where only one data stream is transmitted, and all the frequency or spatial resources at the relay are exploited to provide the maximal diversity gain.
- **Full multiplexing**: The extreme case of $K = M$ or of $\beta = 1$, where the maximal number of interference-free data streams able to be supported in the system are transmitted.
- **Hybrid diversity** and **multiplexing**: The cases with $1 < K < M$, which correspond to $0 < \beta < 1$.

It is worth noting that, when the source transmits multiple data streams, i.e., when $K > 1$, the performance of the multicarrier systems will depend on the multiplexing method to separate multiple data streams [10]. There are two typical multiplexing methods in multicarrier systems, which are CDM and frequency-division multiplexing (FDM). In CDM, different data streams are conveyed over overlapping subcarriers. In FDM, different data streams are transmitted over nonoverlapping subcarriers. For simplicity, we consider the widely known orthogonal multiplexing in the sequel. In this case, the multicarrier system with FDM is an orthogonal frequency-division multiplexing (OFDM) system.

A. Achievable Rate

We first study the performance in the two extreme cases and then consider the general cases.

1) Full Diversity or Full Multiplexing:

Theorem 1: For a multicarrier two-hop relay system with an interference-free transceiver, when $K = 1$ or $K = M$, the achievable rates per data stream in the S–R and R–D links can be expressed in unified forms as

$$R_\mathcal{S} \geq \log\left(1 + \frac{P_\mathcal{S}}{\sigma_\mathcal{R}^2}\mathcal{M}_p\left(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2\right)\right) \tag{12a}$$

TABLE II
VALUES OF p FOR MULTICARRIER RELAY SYSTEMS

Index	Considered cases		CSI mode	Values of p
①	Full diversity		CSI-R	$p = 1$
②			CSI-SRD	$p \to \infty$
③	Full multiplexing	CDM	CSI-R	$p = -1$
④			CSI-SRD	$p = -1/2$
⑤		FDM	CSI-R	$p \to 0$
⑥			CSI-SRD	$p \to 0$

TABLE III
VALUES OF p FOR MIMO RELAY SYSTEMS

Index	Considered cases	CSI mode	Values of p
ⓘ	Full diversity	CSI-R	$p = 1$
ⓘⓘ		CSI-SRD	$p = 1$
ⓘⓘⓘ	Full multiplexing	CSI-R	$p = -1$
ⓘⓥ		CSI-SRD	$p \to 0$

$$R_{\mathcal{D}} \geq \log\left(1 + \frac{P_{\mathcal{R}}}{\sigma_{\mathcal{D}}^2}\mathcal{M}_p\left(|H_{\mathcal{D}_1}|^2, \ldots, |H_{\mathcal{D}_M}|^2\right)\right) \quad (12b)$$

where $H_{\mathcal{S}_m}$ and $H_{\mathcal{D}_m}$ denote the channel responses on the mth subcarrier in the S–R and R–D links, respectively, and the corresponding value of p is listed in Table II.

Theorem 2: For a MIMO two-hop relay system with an interference-free transceiver, where the source and the destination are equipped with K antennas to transmit or receive K data streams, and the relay is equipped with M antennas to forward these data streams, when $K = 1$ or $K = M$, the achievable rates per data stream in the S–R and R–D links can be expressed in unified forms as

$$R_{\mathcal{S}} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_p\left(\Omega_{\mathcal{S}_1}, \ldots, \Omega_{\mathcal{S}_K}\right)\right) \quad (13a)$$

$$R_{\mathcal{D}} \geq \log\left(1 + \frac{P_{\mathcal{R}}}{\sigma_{\mathcal{D}}^2}\mathcal{M}_p\left(\Omega_{\mathcal{D}_1}, \ldots, \Omega_{\mathcal{D}_K}\right)\right) \quad (13b)$$

where $\Omega_{\mathcal{S}_j}$ and $\Omega_{\mathcal{D}_j}$ denote the jth eigenvalues of $\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}$ and $\boldsymbol{H}_{\mathcal{D}}^H\boldsymbol{H}_{\mathcal{D}}$, respectively, and the corresponding value of p is listed in Table III.

According to the definition of generalized mean in (8), the end-to-end achievable rate per data stream of the DF relay system shown in (6) can be rewritten as

$$R = \mathcal{M}_{-1}(R_{\mathcal{S}}, R_{\mathcal{D}})/2. \quad (14)$$

With the unified expressions for the achievable rates or their lower bounds of the multicarrier and MIMO relay systems, we can readily compare their performance.

In multicarrier relay systems, according to Property 2, for arbitrary channels $\boldsymbol{H}_{\mathcal{S}}$ and $\boldsymbol{H}_{\mathcal{D}}$, we always have

$$R_{\mathcal{S}}^{\textcircled{2}} \geq R_{\mathcal{S}}^{\textcircled{1}} \geq R_{\mathcal{S}}^{\textcircled{6}} = R_{\mathcal{S}}^{\textcircled{5}} \geq R_{\mathcal{S}}^{\textcircled{4}} \geq R_{\mathcal{S}}^{\textcircled{3}} \quad (15a)$$

$$R_{\mathcal{D}}^{\textcircled{2}} \geq R_{\mathcal{D}}^{\textcircled{1}} \geq R_{\mathcal{D}}^{\textcircled{6}} = R_{\mathcal{D}}^{\textcircled{5}} \geq R_{\mathcal{D}}^{\textcircled{4}} \geq R_{\mathcal{D}}^{\textcircled{3}}. \quad (15b)$$

Based on Property 1, from (15a) and (15b), we know that

$$R^{\textcircled{2}} \geq R^{\textcircled{1}} \geq R^{\textcircled{6}} = R^{\textcircled{5}} \geq R^{\textcircled{4}} \geq R^{\textcircled{3}}. \quad (16)$$

TABLE IV
VALUES OF p FOR MULTICARRIER OR MIMO RELAY SYSTEMS

Index	Considered cases		CSI mode	Values of p
ⓐ	Multi-carrier	CDM	CSI-R	$p = -1$
ⓑ			CSI-SRD	$p = -1/2$
ⓒ		FDM	CSI-R	$p \to 0$
ⓓ			CSI-SRD	$p \to 0$
ⓔ	MIMO		CSI-R	$p = -1$
ⓕ			CSI-SRD	$p \to 0$

Similarly, for MIMO relay systems, we always have

$$R^{\textcircled{ii}} = R^{\textcircled{i}} \geq R^{\textcircled{iv}} \geq R^{\textcircled{iii}}. \quad (17)$$

2) Hybrid Diversity and Multiplexing: From Theorems 1 and 2, we can observe that, for both multicarrier and MIMO systems, the achievable rate or its lower bound of each data stream can be expressed as a function of the generalized mean. This result can be extended to the more general cases of hybrid diversity and multiplexing, as shown in the following corollary.

Corollary: For the multicarrier system considered in Theorem 1 or the MIMO system considered in Theorem 2, when the number of data streams satisfies $1 \leq K \leq M$, the achievable rates per data stream in the S–R and R–D links can be expressed in unified forms as

$$R_{\mathcal{S}} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_p\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \quad (18a)$$

$$R_{\mathcal{D}} \geq \log\left(1 + \frac{P_{\mathcal{R}}}{\sigma_{\mathcal{D}}^2}\mathcal{M}_p\left(\Omega_{\mathcal{D}_1}^A, \ldots, \Omega_{\mathcal{D}_k}^A\right)\right) \quad (18b)$$

where the corresponding value of p is listed in Table IV.

In (18a) and (18b), $\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A$ or $\Omega_{\mathcal{D}_1}^A, \ldots, \Omega_{\mathcal{D}_K}^A$ are the eigenvalues of the equivalent channel correlation matrices $\boldsymbol{\Omega}_{\mathcal{S}}$ and $\boldsymbol{\Omega}_{\mathcal{D}}$, respectively, where $\boldsymbol{\Omega}_{\mathcal{S}} = \boldsymbol{W}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}\boldsymbol{W}_{\mathcal{S}} \in \mathbb{C}^{K\times K}$, and $\boldsymbol{\Omega}_{\mathcal{D}} = \boldsymbol{W}_{\mathcal{D}}^H\boldsymbol{H}_{\mathcal{D}}\boldsymbol{H}_{\mathcal{D}}^H\boldsymbol{W}_{\mathcal{D}} \in \mathbb{C}^{K\times K}$.

When we consider the optimal transceivers that maximize the received SNR, the values of $\Omega_{\mathcal{T}_1}^A, \ldots, \Omega_{\mathcal{T}_K}^A, \mathcal{T} = \mathcal{S}, \mathcal{D}$ differ in the cases of Table IV, as shown in the following.

- The values of $\Omega_{\mathcal{T}_1}^A, \ldots, \Omega_{\mathcal{T}_K}^A$ are the eigenvalues of $\boldsymbol{\Omega}_{\mathcal{T}} = \boldsymbol{C}^H\boldsymbol{H}_{\mathcal{T}}^H\boldsymbol{H}_{\mathcal{T}}\boldsymbol{C}$ for case ⓐ, where $\boldsymbol{C}$ is an $M \times K$ unitary matrix denoting the orthogonal spreading sequences employed by CDM.
- The values of $\Omega_{\mathcal{T}_1}^A, \ldots, \Omega_{\mathcal{T}_K}^A$ are the K largest variables of $\{|H_{\mathcal{T}_1}|^2, \ldots, |H_{\mathcal{T}_M}|^2\}$ for cases ⓑ and ⓓ.
- The values of $\Omega_{\mathcal{T}_1}^A, \ldots, \Omega_{\mathcal{T}_K}^A$ are

$$\Omega_{\mathcal{T}_j}^A = \begin{cases} \frac{1}{N_1}\sum_{n=0}^{N_1-1}|H_{\mathcal{T}_{nK+j}}|^2, & j \in [1, M - KN_2] \\ \frac{1}{N_2}\sum_{n=0}^{N_2-1}|H_{\mathcal{T}_{nK+j}}|^2, & j \in (M - KN_2, K] \end{cases} \quad (19)$$

  for case ⓒ, where $N_1 = \lceil M/K \rceil$, and $N_2 = \lfloor M/K \rfloor$.
- The values of $\Omega_{\mathcal{T}_1}^A, \ldots, \Omega_{\mathcal{T}_K}^A$ are the eigenvalues of $\boldsymbol{\Omega}_{\mathcal{T}} = \boldsymbol{H}_{\mathcal{T}}^H\boldsymbol{H}_{\mathcal{T}}$ for cases ⓔ and ⓕ.

B. Spectral Efficiency

From Corollary 1, we know that, upon substituting the joint probability density function (pdf) of $\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A$ and $\Omega_{\mathcal{D}_1}^A, \ldots, \Omega_{\mathcal{D}_K}^A$ into (7), we can obtain the average spectral efficiency of the two-hop relay systems. However, the analysis in [11] indicates that, when $K > 2$, it is intractable to derive a closed-form expression for the average spectral efficiency. Fortunately, as $K, M \to \infty$ with $K/M \to \beta$, with the aid of the random matrix theory, we can derive a closed-form expression for the asymptotic spectral efficiency, which converges in mean square to the average spectral efficiency. As will be shown later by our simulation results, the asymptotic spectral efficiency is close to the average spectral efficiency even with finite values of K and M. In the sequel, we will analyze the asymptotic spectral efficiency for general two-hop DF relay systems.

Corollary 2: For the multicarrier system considered in Theorem 1 or the MIMO system considered in Theorem 2, when the number of data streams satisfies $1 \le K \le M$, the asymmetric spectral efficiency can be expressed in a unified form as

$$\eta = \frac{\beta}{2}\log\left(1 + \frac{P}{\sigma_{\mathcal{R}}^2 + \sigma_{\mathcal{D}}^2} O\right) \tag{20}$$

where

$$O = \left(\int x^p f_\Omega(x) dx\right)^{\frac{1}{p}} \tag{21}$$

and $f_\Omega(x)$ is the pdf of $\Omega_{\mathcal{T}_1}^{\mathcal{A}}, \ldots, \Omega_{\mathcal{T}_K}^{\mathcal{A}}$.

Specifically, when Rayleigh fading channels are assumed, the values of O corresponding to the different cases listed in Table IV are as follows:

$$O^{ⓐ} = \frac{\beta}{(1-\beta)\mathcal{H}(1-\beta)} \tag{22a}$$

$$O^{ⓑ} = \left(\sum_{j=1}^{K} \frac{\sum_{i=0}^{M-j}\binom{M-j}{i}(-1)^i\sqrt{\frac{\pi}{i+j}}}{B(j, M-j+1)K}\right)^{-2} \tag{22b}$$

$$O^{ⓒ} = \rho\frac{\exp(\psi(N_1))}{N_1} + (1-\rho)\frac{\exp(\psi(N_2))}{N_2} \tag{22c}$$

$$O^{ⓓ} = \exp\left(\sum_{j=1}^{K} \frac{\sum_{i=0}^{M-j}\binom{M-j}{i}\frac{(-1)^{i+1}\ln(i+j)+\gamma_E}{i+j}}{B(j, M-j+1)K}\right) \tag{22d}$$

$$O^{ⓔ} = MK \tag{22e}$$

$$O^{ⓕ} = \lim_{p\to 0} Ma\sqrt[p]{a^{-1}{}_2F_1\left(1-p, \frac{3}{2}; 3; \frac{a-b}{b}\right)} \tag{22f}$$

where

- $\mathcal{H}(x) = \mathcal{G}^{-1}(x)$ is the inverse function of $\mathcal{G}(x)$, $\mathcal{G}(x) = (1/x)\exp(1/x)\mathrm{Ei}(-1/x)$, and $\mathrm{Ei}(x)$ is the exponential integral function [9];
- $\rho = M/K - \lfloor M/K \rfloor$;
- $B(x, y)$ is the beta function [12];
- $\gamma_E = 0.577216\ldots$ is the Euler–Mascheroni constant [13];
- $\psi(x)$ is the digamma function [12];
- ${}_2F_1(\mu, \nu; \gamma; z)$ is the Gauss hypergeometric function [12].

IV. Proof of the Main Results

Here, we prove the results stated in Section III. Along with our proofs, we also show the corresponding transceivers to achieve these results.

A. Proof of Theorem 1

1) Full Diversity: When the source has no CSI but the relay has the CSI (i.e., the case of CSI-R), it is well known that the optimal precoder at the source is *repetition diversity*, and the optimal receiver at the relay is the *maximal-ratio combining* (MRC) [6], [14], i.e., $\boldsymbol{w}_{\mathcal{S}_1} = 1/\sqrt{M}[1, \ldots, 1]^T$, and $\boldsymbol{w}_{\mathcal{R}r_1}^H = \boldsymbol{w}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}}^H$. Upon substituting into (4a), the SNR of the single data stream is $\gamma_{\mathcal{S}_1} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\sum_{m=1}^{M}|H_{\mathcal{S}_m}|^2/M = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\mathcal{M}_1(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2)$. From (5a), the achievable rate per data stream in the S–R link is

$$R_{\mathcal{S}}^{①} = \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_1\left(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2\right)\right) \tag{23}$$

which is related to the generalized mean associated with $p = 1$.

When both the source and the relay have the CSI (i.e., the case of CSI-SRD), the optimal precoder at the source is the *selection diversity* [6], [15]. That is to say, the signal is transmitted and received only at the subcarrier with the highest channel gain. Therefore, the SNR of the single data stream is $\gamma_{\mathcal{S}_1} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\max\{|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2\} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\mathcal{M}_\infty(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2)$. Then, the achievable rate per data stream is

$$R_{\mathcal{S}}^{②} = \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_\infty\left(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2\right)\right) \tag{24}$$

which is related to the generalized mean associated with $p \to \infty$.

2) Full Multiplexing: In the multicarrier systems using CDM and operated under the CSI-R mode, the optimal precoder uses orthogonal spreading sequences (e.g., Walsh–Hadamard codes) to spread the transmit signals, and the optimal interference-free receiver is the zero-forcing (ZF) detector. Correspondingly, we have $\boldsymbol{W}_{\mathcal{S}} = \boldsymbol{C}$ and $\boldsymbol{W}_{\mathcal{R}r}^H = \boldsymbol{C}^H\boldsymbol{H}_{\mathcal{S}}^{-1}$, where $\boldsymbol{C}$ is an $M \times M$ unitary matrix whose (m, k)th element satisfies $|C_{m,k}|^2 = 1/M$ [16]. Upon substituting into (4a), it is not hard to obtain that $\gamma_{\mathcal{S}_1} = \cdots = \gamma_{\mathcal{S}_K} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)(\sum_{m=1}^{M}|H_{\mathcal{S}_m}|^{-2}/M)^{-1} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\mathcal{M}_{-1}(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2)$. Therefore, the achievable rate per data stream in the S–R link is

$$R_{\mathcal{S}}^{③} = \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_{-1}\left(|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2\right)\right) \tag{25}$$

which is related to the generalized mean associated with $p = -1$.

In the case of CSI-SRD, the optimal transceiver is a *balanced equalization* [17], where both the transmitter and the receiver, respectively, mitigate one half of the interference, i.e., $\boldsymbol{W}_{\mathcal{S}} = a\boldsymbol{H}_{\mathcal{S}}^{-1/2}\boldsymbol{C}$ and $\boldsymbol{W}_{\mathcal{R}r}^H = a^{-1}\boldsymbol{C}^H\boldsymbol{H}_{\mathcal{S}}^{-1/2}$, where a is a scaling factor to meet the transmit power constraint. After substituting into (4a), we have $\gamma_{\mathcal{S}_1} = \cdots = \gamma_{\mathcal{S}_K} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)$

$(\sum_{m=1}^{M}|H_{\mathcal{S}_m}|^{-1}/M)^{-2}=(P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\mathcal{M}_{-1/2}(|H_{\mathcal{S}_1}|^2,\ldots,|H_{\mathcal{S}_M}|^2)$. Then, the achievable rate per data stream becomes

$$R_{\mathcal{S}}^{④}=\log\left(1+\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_{-\frac{1}{2}}\left(|H_{\mathcal{S}_1}|^2,\ldots,|H_{\mathcal{S}_M}|^2\right)\right) \tag{26}$$

which is related to the generalized mean associated with $p=-1/2$.

In the multicarrier systems using FDM, i.e., OFDM systems, each data stream is transmitted at only one subcarrier. In the case of CSI-R, the SNR of the kth symbol is $\gamma_{\mathcal{S}_k}=(P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)|H_{\mathcal{S}_k}|^2$. According to Property 3, we have $\sum_{k=1}^{M}\log(1+\gamma_{\mathcal{S}_k})/M\geq\log(1+\mathcal{M}_0(\gamma_{\mathcal{S}_1},\ldots,\gamma_{\mathcal{S}_M}))$. Therefore, we obtain a lower bound of the achievable rate per data stream as follows:

$$R_{\mathcal{S}}^{⑤}\geq\log\left(1+\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_0\left(|H_{\mathcal{S}_1}|^2,\ldots,|H_{\mathcal{S}_M}|^2\right)\right) \tag{27}$$

which is related to the generalized mean associated with $p\to 0$. Furthermore, it is not difficult to find that the equality of (27) holds when $\gamma_{\mathcal{S}_k}\gg 1$. This indicates that the lower bound is tight at a high SNR level.

In the case of CSI-SRD, the optimal precoder uses the *water-filling*-assisted power allocation. It reduces to equal power allocation in the high-SNR region, but outperforms the equal power allocation in a general SNR level [15]. Considering that, when CSI is not available at the source, the optimal power allocation is the equal power allocation, we have $R_{\mathcal{S}}^{⑥}\geq R_{\mathcal{S}}^{⑤}$. Then, we obtain

$$R_{\mathcal{S}}^{⑥}\geq\log\left(1+\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_0\left(|H_{\mathcal{S}_1}|^2,\ldots,|H_{\mathcal{S}_M}|^2\right)\right) \tag{28}$$

which is related to the generalized mean associated with $p\to 0$. In (28), the equality becomes more declared at high SNR levels.

Considering (23)–(28), we obtain (12a) and Table II, where all the equalities hold at a high SNR level. Following the same way, we can derive the achievable rate per data stream in the R–D link for different cases and obtain (12b).

B. Proof of Theorem 2

1) Full Diversity: In the considered MIMO system, the number of antennas at the source is identical to the number of data streams. In the case of "full diversity," $K=1$, i.e., the source only has one antenna. Then, the S–R link corresponds to a single-input–multiple-output system, and the channel matrix $\boldsymbol{H}_{\mathcal{S}}$ reduces to a vector, denoted by $\boldsymbol{h}_{\mathcal{S}}$. Since the source only has one antenna, the preprocessing of the source will not affect the performance in the S–R link. Consequently, the systems with and without the CSI available at the source achieve the same performance.

Since the relay has the CSI, the optimal receiver is the MRC, i.e., $\boldsymbol{w}_{\mathcal{R}r_1}^H=\boldsymbol{h}_{\mathcal{S}}^H$. Upon substituting into (4a), the SNR of the single data stream is $\gamma_{\mathcal{S}_1}=(P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\boldsymbol{h}_{\mathcal{S}}^H\boldsymbol{h}_{\mathcal{S}}=(P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\Omega_{\mathcal{S}_1}$, where $\Omega_{\mathcal{S}_1}=\boldsymbol{h}_{\mathcal{S}}^H\boldsymbol{h}_{\mathcal{S}}$. Since $\Omega_{\mathcal{S}_1}=\mathcal{M}_1(\Omega_{\mathcal{S}_1})$, the achievable rate per data stream in the S–R link can be expressed as

$$R_{\mathcal{S}}^{ⓘ}=R_{\mathcal{S}}^{ⓘⓘ}=\log\left(1+\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_1(\Omega_{\mathcal{S}_1})\right). \tag{29}$$

Explicitly, it is related to the generalized mean associated with $p=1$.

2) Full Multiplexing: In the case of CSI-R, the optimal precoder at the source is $\boldsymbol{W}_{\mathcal{S}}=\boldsymbol{I}$, and the optimal receiver at the relay is the ZF detector $\boldsymbol{W}_{\mathcal{R}r}^H=(\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}})^{-1}\boldsymbol{H}_{\mathcal{S}}^H$ [18]. Upon substituting into (4a), the SNR of the kth data stream is obtain as [18]

$$\gamma_{\mathcal{S}_k}=\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\frac{1}{\boldsymbol{e}_k^T\left(\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}\right)^{-1}\boldsymbol{e}_k}=\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\frac{1}{\sum_{m=1}^{M}|V_{m,k}|^2\Omega_{\mathcal{S}_M}^{-1}} \tag{30}$$

where $\boldsymbol{e}_k$ is a basis vector, whose kth entry is one and all the other entries are zeros; $\Omega_{\mathcal{S}_m}$ is the mth eigenvalue of $\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}$; $V_{m,k}$ is the mth entry of $\boldsymbol{V}$; and $\boldsymbol{V}$ is the eigenvector matrix of $\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}$.

According to Properties 2 and 3, we have $\sum_{k=1}^{M}\log(1+\gamma_{\mathcal{S}_k})/M\geq\log(1+\mathcal{M}_{-1}(\gamma_{\mathcal{S}_1},\ldots,\gamma_{\mathcal{S}_M}))$. In (30), $\boldsymbol{V}$ is a unitary matrix and then satisfies $\boldsymbol{V}^H\boldsymbol{V}=\boldsymbol{I}$, i.e., $\sum_{k=1}^{M}|V_{m,k}|^2=1$. Hence, we have

$$\begin{aligned}\mathcal{M}_{-1}(\gamma_{\mathcal{S}_1},\ldots,\gamma_{\mathcal{S}_M})&=\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\frac{1}{\frac{1}{M}\sum_{k=1}^{M}\sum_{m=1}^{M}|V_{m,k}|^2\Omega_{\mathcal{S}_M}^{-1}}\\&=\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\frac{1}{\frac{1}{M}\sum_{m=1}^{M}\Omega_{\mathcal{S}_M}^{-1}}\\&=\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_{-1}(\Omega_{\mathcal{S}_1},\ldots,\Omega_{\mathcal{S}_M}).\end{aligned} \tag{31}$$

Then, the achievable rate per data stream in the S–R link satisfies

$$R_{\mathcal{S}}^{ⓘⓘⓘ}\geq\log\left(1+\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_{-1}\left(\Omega_{\mathcal{S}_1},\ldots,\Omega_{\mathcal{S}_M}\right)\right) \tag{32}$$

which is a function of the generalized mean associated with $p=-1$. When $|V_{m,k}|^2=1/M$ or $M,K\to\infty$, it is not hard to obtain $\gamma_{\mathcal{S}_1}=\cdots=\gamma_{\mathcal{S}_M}$. Then, the equality in (32) holds.

In the case of CSI-SRD, it is well known that the optimal precoder at the source is the *singular value decomposition (SVD) of the channel matrix* in conjunction with the *water-filling* power allocation. In the high-SNR region, the water-filling power allocation reduces to the equal power allocation, and the optimal transceiver in the S–R link can be expressed as $\boldsymbol{W}_{\mathcal{S}}=\boldsymbol{V}$ and $\boldsymbol{W}_{\mathcal{R}r}^H=\boldsymbol{U}^H$, where $\boldsymbol{H}_{\mathcal{S}}=\boldsymbol{U}\boldsymbol{\Sigma}_{\mathcal{S}}\boldsymbol{V}^H$ is the SVD of $\boldsymbol{H}_{\mathcal{S}}$. Upon substituting into (4a), the SNR of the kth symbol in the S–R link is obtain as $\gamma_{\mathcal{S}_k}=P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2\Omega_{\mathcal{S}_k}$, where $\Omega_{\mathcal{S}_k}$ is the kth eigenvalue of $\boldsymbol{H}_{\mathcal{S}}^H\boldsymbol{H}_{\mathcal{S}}$. In the general SNR level, we have $R_{\mathcal{S}}^{ⓘⓥ}\ \sum_{k=1}^{M}\log(1+P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2\Omega_{\mathcal{S}_k})/M$. According to Property 3, we obtain a lower bound of the achievable rate per data stream as follows:

$$R_{\mathcal{S}}^{ⓥ}\geq\log\left(1+\frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2}\mathcal{M}_0(\Omega_{\mathcal{S}_1},\ldots,\Omega_{\mathcal{S}_M})\right) \tag{33}$$

which is a function of the generalized mean associated with $p \to 0$. Here, the equality is achieved as the SNR tends to infinity.

Considering (29), (32), and (33), we obtain (13a) and Table III. Following the same way, we can derive the achievable rate per data stream in the R–D link in different cases and obtain (13b).

C. Proof of Corollary 1

1) CDM Multicarrier Systems: In the case of CSI-R, the optimal precoding is to use spreading sequences, i.e., $\boldsymbol{W}_{\mathcal{S}} = \boldsymbol{C}$, where $\boldsymbol{C} \in \mathbb{C}^{M\times K}$, and the optimal receiver to achieve the maximal SNR is the symbol-level equalizer [10], i.e., $\boldsymbol{W}_{\mathcal{R}r}^H = (\boldsymbol{C}^H \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}} \boldsymbol{C})^{-1} \boldsymbol{C}^H \boldsymbol{H}_{\mathcal{S}}^H$. Upon substituting into (4a), we obtain the SNR of the kth data stream as follows:

$$\begin{aligned}\gamma_{\mathcal{S}_k} &= \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \frac{1}{\boldsymbol{e}_k^T \left(\boldsymbol{C}^H \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}} \boldsymbol{C}\right)^{-1} \boldsymbol{e}_k} \\ &= \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \frac{1}{\sum_{j=1}^{K} |V_{j,k}^A|^2 (\Omega_{\mathcal{S}_j}^A)^{-1}} \end{aligned} \tag{34}$$

where $\Omega_{\mathcal{S}_j}^A$ is the jth eigenvalue of $\boldsymbol{\Omega}_{\mathcal{S}} = \boldsymbol{C}^H \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}} \boldsymbol{C}$, and $V_{j,k}^A$ is the (j,k)th entry of the eigenvector matrix of $\boldsymbol{\Omega}_{\mathcal{S}}$.

Following the similar derivation for (32), we obtain

$$R_{\mathcal{S}}^{ⓐ} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \mathcal{M}_{-1}\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \tag{35}$$

which is related to the generalized mean associated with $p = -1$. In (35), the equality holds when $|V_{j,k}^A|^2 = 1/K$.

In the case of CSI-SRD, *selection diversity* is optimal to maximize the received SNR [10], [15]. When $K > 1$, the source needs to choose K subcarriers from the M subcarriers with highest channel gains. Then, we have

$$\left\{\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right\} = \max_K \left\{|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2\right\} \tag{36}$$

where $\max_K\{\cdot\}$ denotes choosing the K largest variables from all the variables.

After choosing K desired subcarriers, the multicarrier CDM system turns into a system with full multiplexing. Therefore, by using the *balanced equalization* again and following the similar derivation for $R_{\mathcal{S}}^{④}$, we can obtain the achievable rate per data stream in the S–R link as

$$R_{\mathcal{S}}^{ⓑ} = \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \mathcal{M}_{-\frac{1}{2}}\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \tag{37}$$

which is a function of the generalized mean associated with $p = -1/2$.

2) FDM Multicarrier Systems: When $K < M$, there exist redundant subcarriers. To exploit the redundant frequency resource to obtain diversity, here, we consider a FDM system that uses the redundant subcarriers to change the power response of the equivalent channel for the kth symbol, i.e., $\Omega_{\mathcal{S}_k}^A$. Then, the SNR of the kth symbol becomes $\gamma_{\mathcal{S}_k} = (P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2)\Omega_{\mathcal{S}_k}^A$ [10].

In the case of CSI-R, the analysis in [16] indicates that the optimal precoder should employ *repetition diversity* to maximize the diversity gain. This is equivalent to design the precoder to combine maximal number of subcarriers with maximal spacing and with equal gain weighting. When M is divisible by K, the optimal precoder combines M/K subcarriers with an interval of K subcarriers, which gives rise to the power response of the equivalent channel after the precoding as $\Omega_{\mathcal{S}_j}^A = K/M \sum_{n=0}^{M/K-1} |H_{\mathcal{S}_{nK+j}}|^2$. In the general cases, the optimal form of $\Omega_{\mathcal{S}_j}^A$ is (19) [16].

Following the similar derivation for $R_{\mathcal{S}}^{⑤}$, we obtain

$$R_{\mathcal{S}}^{ⓒ} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \mathcal{M}_{0}\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \tag{38}$$

where the equality is declared, when the SNR is infinity.

In the case of CSI-SRD, the optimal precoder will choose the K subcarriers with highest gains. Therefore, the eigenvalues of equivalent channel correlation matrix are the same as (36). After selecting the K subcarriers and deriving the achievable rate in the same way as for $R_{\mathcal{S}}^{⑥}$, the achievable rate per data stream in the S–R link becomes

$$R_{\mathcal{S}}^{ⓓ} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \mathcal{M}_{0}\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \tag{39}$$

which is related to the generalized mean associated with $p \to 0$, and where the equality becomes more declared as the SNR increases.

3) MIMO Systems: For the MIMO system considered in Theorem 2, in the case of CSI-R, for arbitrary number of data streams with $1 \leq K \leq M$, it is not difficult to show that the equal power allocation and the ZF detector, i.e., $\boldsymbol{W}_{\mathcal{S}} = \boldsymbol{I}_K$ and $\boldsymbol{W}_{\mathcal{R}r}^H = (\boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}})^{-1} \boldsymbol{H}_{\mathcal{S}}^H \in \mathbb{C}^{K\times M}$, are still the optimal interference-free transceiver to maximize the SNR. Following the similar derivation for (32), we have

$$R_{\mathcal{S}}^{ⓔ} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \mathcal{M}_{-1}\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \tag{40}$$

which is a function of the generalized mean associated with $p = -1$, and where $\Omega_{\mathcal{S}_j}^A$ is the jth eigenvalue of $\boldsymbol{\Omega}_{\mathcal{S}} = \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}}$.

In the case of CSI-SRD, again considering the SVD beamforming with water-filling power allocation and following the similar derivation for $R_{\mathcal{S}}^{(iv)}$, we can obtain the achievable rate per data stream as follows:

$$R_{\mathcal{S}}^{ⓕ} \geq \log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} \mathcal{M}_{0}\left(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_k}^A\right)\right) \tag{41}$$

which is a function of the generalized mean associated with $p \to 0$ and where $\Omega_{\mathcal{S}_j}^A$ is the jth eigenvalue of $\boldsymbol{\Omega}_{\mathcal{S}} = \boldsymbol{V}^H \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}} \boldsymbol{V} = \boldsymbol{\Sigma}_{\mathcal{S}}^H \boldsymbol{\Sigma}_{\mathcal{S}}$, and $\boldsymbol{H}_{\mathcal{S}} = \boldsymbol{U}\boldsymbol{\Sigma}_{\mathcal{S}}\boldsymbol{V}^H$ is the SVD of $\boldsymbol{H}_{\mathcal{S}}$. Considering (35) and (37)–(41), we obtain (18a) and Table IV. Following the same way, we can derive the achievable rate per data stream in the R–D link in different cases and obtain (18b).

D. Proof of Corollary 2

As $K, M \to \infty$ with $K/M \to \beta$, the asymptotic achievable rates per data stream in the S–R and R–D links become $R_{\mathcal{S}} = (1 + P_{\mathcal{S}}/\sigma_{\mathcal{R}}^2 O)$ and $R_{\mathcal{D}} = (1 + P_{\mathcal{R}}/\sigma_{\mathcal{D}}^2 O)$, respectively, where $O = \lim_{M,K\to\infty} \mathcal{M}_p(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A) = \lim_{M,K\to\infty} \mathcal{M}_p(\Omega_{\mathcal{D}_1}^A, \ldots, \Omega_{\mathcal{D}_K}^A)$.[1] According to the Kolmogorov's strong law of large numbers [19], we obtain (21).

Upon substituting the achievable rates into (6) and then into (7), we obtain the asymptotic spectral efficiency as

$$\eta = \beta \frac{\log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} O\right) \log\left(1 + \frac{P_{\mathcal{R}}}{\sigma_{\mathcal{D}}^2} O\right)}{\log\left(1 + \frac{P_{\mathcal{S}}}{\sigma_{\mathcal{R}}^2} O\right) + \log\left(1 + \frac{P_{\mathcal{R}}}{\sigma_{\mathcal{D}}^2} O\right)}. \tag{42}$$

Given the overall transmit power per symbol $P = P_{\mathcal{S}} + P_{\mathcal{R}}$ in the three-node system, from (42), it is not hard to derive the optimal power allocation that maximizes the asymptotic spectral efficiency as follows:

$$P_{\mathcal{S}} = \frac{\sigma_{\mathcal{R}}^2}{\sigma_{\mathcal{R}}^2 + \sigma_{\mathcal{D}}^2} P$$
$$P_{\mathcal{R}} = \frac{\sigma_{\mathcal{D}}^2}{\sigma_{\mathcal{R}}^2 + \sigma_{\mathcal{D}}^2} P. \tag{43}$$

Substituting (43) into (42), we obtain the maximal asymptotic spectral efficiency shown in (20).

1) CDM Multicarrier Systems: In the case of CSI-R, the correlation matrix of the equivalent channel in the S–R link is $\boldsymbol{\Omega}_{\mathcal{S}} = \boldsymbol{C}^H \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}} \boldsymbol{C}$. Therefore, the CDM systems with different spreading sequences will achieve different performance.

Since the analysis in [16] indicates that the orthogonal spreading sequences uniformly distributed over the manifold of complex matrices can provide the maximal achievable rate, we will consider this kind of spreading sequences. It is shown in [9, Example 2.51] that, when $\boldsymbol{C}$ is a random matrix uniformly distributed over the manifold of $M \times K$ complex matrices satisfying $\boldsymbol{C}^H \boldsymbol{C} = \boldsymbol{I}$ and $\boldsymbol{H}_{\mathcal{S}}$ is a random matrix independent of $\boldsymbol{C}$, as $K, M \to \infty$ with $K/M \to \beta$, the empirical distribution of the eigenvalues of $\boldsymbol{C}^H \boldsymbol{H}_{\mathcal{S}}^H \boldsymbol{H}_{\mathcal{S}} \boldsymbol{C}$ converges with a probability of one to a distribution that satisfies

$$\mathcal{G}\left(\frac{\beta}{1-\beta} \int x^{-1} f_{\Omega}(x) dx\right) = 1 - \beta \tag{44}$$

where $\mathcal{G}(x) = (1/x)\exp(1/x)\mathrm{Ei}(-1/x)$ in Rayleigh fading channels [9]. Substituting (44) into (21) with $p = -1$, we have (22a).

In the case of CSI-SRD, from (36), we know that $\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A$ are the K largest variables of $|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2$. To derive the pdf of $\Omega_{\mathcal{S}_j}^A$, we first develop the pdf of the random variables of $\{\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A\}$ in a descending order, denoted by $\{\Omega_1^{\downarrow}, \ldots, \Omega_K^{\downarrow}\}$, where $\Omega_1^{\downarrow} \geq \cdots \geq \Omega_K^{\downarrow}$. Let $f(x)$ and $F(x)$ denote the pdf of $|H_{\mathcal{S}_j}|^2$ and the corresponding cumulative distribution function, respectively. From the theory of order statistics in [20], we can derive the pdf of the jth largest variable of $|H_{\mathcal{S}_1}|^2, \ldots, |H_{\mathcal{S}_M}|^2$, i.e., the pdf of $\Omega_j^{\downarrow}$ as

$$f_{\Omega_j^{\downarrow}}(x) = \frac{F^{M-j}(x)\,(1-F(x))^{j-1} f(x)}{B(j, M-j+1)}. \tag{45}$$

Since the variables of $\{\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A\}$ are unordered and in the same variable set as $\{\Omega_1^{\downarrow}, \ldots, \Omega_K^{\downarrow}\}$, we know that $\Omega_{\mathcal{S}_1^A}, \ldots, \Omega_{\mathcal{S}_K}^A$ have the same pdf, which is $f_{\Omega}(x) = \sum_{j=1}^{K} f_{\Omega_j^{\downarrow}}(x)/K$. In Rayleigh fading channels, $f(x) = e^{-x}$ and $F(x) = 1 - e^{-x}$, then we have

$$f_{\Omega}(x) = \sum_{j=1}^{K} \frac{\sum_{i=0}^{M-j} \binom{M-j}{i} (-1)^i e^{-(i+j)x}}{B(j, M-j+1)K}. \tag{46}$$

Upon substituting (46) into (21) with $p = -1/2$, for the case of CSI-SRD, we have

$$O^{\textcircled{b}} = \left(\sum_{j=1}^{K} \frac{\sum_{i=0}^{M-j} \binom{M-j}{i} (-1)^i \int_0^{\infty} x^{-\frac{1}{2}e^{-(i+j)x}} dx}{B(j, M-j+1)K}\right)^{-2}. \tag{47}$$

According to [12, (3.381.4)], we know that $\int_0^{\infty} x^{-1/2} e^{-nx} dx = n^{-1/2}\Gamma(1/2) = \sqrt{\pi/n}$. Substituting it into (47), finally we obtain (22b).

2) FDM Multicarrier Systems: In the multicarrier system using FDM, from Table IV, we know that the achievable rate depends on $\mathcal{M}_0(\Omega_{\mathcal{S}_1}^A, \ldots, \Omega_{\mathcal{S}_K}^A)$ for both cases of CSI-R and CSI-SRD. From the definition of generalized mean with order zero in Table I, we can derive

$$\lim_{p\to 0} O = \lim_{p\to 0} \sqrt[p]{\int x^p f_{\Omega}(x) dx} \tag{48a}$$
$$= \exp\left(\int \ln x f_{\Omega}(x) dx\right). \tag{48b}$$

In the case of CSI-R, the eigenvalues of equivalent channel correlation matrix in the S–R link is shown in (19). In Rayleigh fading channels, from [21], we know that, for $1 \leq j \leq M - KN_2$, the pdf of the eigenvalue $\Omega_{\mathcal{S}_j}^A$ is $f_{\Omega_1}(x) = N_1^{N_1} x^{N_1-1} e^{-N_1 x}/\Gamma(N_1)$, whereas for $M - KN_2 < j \leq K$, the pdf of $\Omega_{\mathcal{S}_j}^A$ is $f_{\Omega_2}(x) = N_2^{N_2} x^{N_2-1} e^{-N_2 x}/\Gamma(N_2)$, where $\Gamma(x)$ is the Gamma function. Upon substituting these pdf functions into (48b), we obtain

$$O^{\textcircled{c}} = \rho \exp\left(\int_0^{\infty} \frac{N_1^{N_1}}{\Gamma(N_1)} x^{N_1-1} e^{-N_1 x} \ln x dx\right)$$
$$+ (1-\rho)\exp\left(\int_0^{\infty} \frac{N_2^{N_2}}{\Gamma(N_2)} x^{N_2-1} e^{-N_2 x} \ln x dx\right). \tag{49}$$

From [12, (4.352.1)], we know $\int_0^{\infty} x^{N-1} e^{-Nx} \ln x dx = \Gamma(N)/N^N(\psi(N) - \ln N)$. Then, (49) becomes (22c).

[1]It is reasonable to assume that $\boldsymbol{\Omega}_{\mathcal{S}}$ and $\boldsymbol{\Omega}_{\mathcal{D}}$ have the same statistic characteristics, i.e., $\Omega_{\mathcal{S}_j}^A$ and $\Omega_{\mathcal{D}_j}^A$ have the same pdf.

In the case of CSI-SRD, substituting the pdf of $\Omega^A_{\mathcal{S}_1}, \ldots, \Omega^A_{\mathcal{S}_K}$ shown in (46) into (48b), we obtain

$$O^{ⓓ} = \exp\left(\sum_{j=1}^{K} \frac{\sum_{i=0}^{M-j} \binom{M-j}{i}(-1)^i \int_0^\infty \ln x e^{-(i+j)x} dx}{B(j, M-j+1)K}\right). \quad (50)$$

From [12, (4.331.1)], we know that $\int_0^\infty e^{-nx} \ln x dx = -(\gamma_E + \ln n)/n$. Substituting it into (50), we obtain (22d).

3) MIMO Systems: In MIMO systems, from previous analysis, we know that no matter if the source has the CSI or not, $\Omega^A_{\mathcal{S}_1}, \ldots, \Omega_{\mathcal{S}_K}$ are the eigenvalues of $\boldsymbol{H}^H_{\mathcal{S}} \boldsymbol{H}_{\mathcal{S}}$. In the considered MIMO system, the entries of $M \times K$ channel matrix $\boldsymbol{H}_{\mathcal{S}}$ are assumed to be i.i.d. random variables with zero mean and unit variance. Hence, the entries of $\boldsymbol{H}_{\mathcal{S}}/\sqrt{M}$ are also zero-mean i.i.d. random variables but with variance $1/M$. From [9], we know that as $K, M \to \infty$ with $K/M \to \beta$, the empirical distribution of the eigenvalues of $\boldsymbol{H}_{\mathcal{S}} \boldsymbol{H}^H_{\mathcal{S}}/M$ (i.e., $\Omega^A_{\mathcal{S}_1}/M, \ldots, \Omega^A_{\mathcal{S}_K}/M$) converges almost surely to a distribution whose pdf is

$$f_\Omega(x) = \frac{\sqrt{(x-a)^+(b-x)^+}}{2\pi\beta x} \quad (51)$$

where $a = (1-\sqrt{\beta})^2$ and $b = (1+\sqrt{\beta})^2$.

In the case of CSI-R, from (40), we know that the achievable rate is related to the generalized mean of $\Omega^A_{\mathcal{S}_1}, \ldots, \Omega^A_{\mathcal{S}_K}$ with $p = -1$. Substituting (51) into (21) with $p = -1$, we have

$$O^{ⓔ} = \frac{M}{\int_a^b \frac{\sqrt{(x-a)(b-x)}}{2\pi\beta x^2} dx}. \quad (52)$$

Upon substituting [12, (3.197.8)] into (51), we obtain (21).

In the case of CSI-SRD, from Table IV, we know that the achievable rate depends on $\mathcal{M}_0(\Omega^A_{\mathcal{S}_1}, \ldots, \Omega^A_{\mathcal{S}_K})$. If we substitute (51) into (48b), the closed-form asymptotic spectral efficiency cannot be obtained since there is no closed-form expression for $\int \ln x f_\Omega(x) dx$. By contrast, there is a closed-form expression for $\int x^p f_\Omega(x) dx$ from [12, (3.197.8)]. Therefore, by substituting (51) into (48) instead, we can derive the closed-from asymptotic spectral efficiency as follows:

$$O^{ⓕ} = M \lim_{p \to 0} \sqrt[p]{\int x^p f_\Omega(x) dx}. \quad (53)$$

Substituting [12, (3.197.8)] into (53), we obtain (22).

V. Simulation and Numerical Results

Here, we validate previous analysis by comparing the derived asymptotic spectral efficiency with the average spectral efficiency obtained through simulations with finite numbers of K and M. The channels on different subcarriers or antennas are assumed to be i.i.d. Rayleigh fading channels. In the simulation, we consider that noise at the relay and the destination has the same variance of $\sigma^2_{\mathcal{R}} = \sigma^2_{\mathcal{D}} = \sigma^2$. The transmit power is equally allocated to each symbol at the source and the relay, i.e., $P_{\mathcal{S}} = P_{\mathcal{R}} = P/2$. The SNR per symbol is P/σ^2. All simulation results are averaged over 5000 channel realizations.

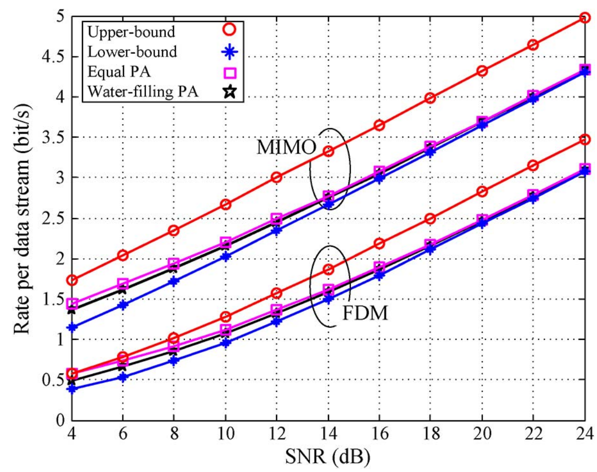

Fig. 1. Achievable rate per data stream versus SN in the case of full multiplexing.

For the cases of ⑤ and ⑥ in Theorem 1, and ⓘⓥ in Theorem 2, the generalized means are not accurate expressions of achievable rates but the lower bounds based on two approximations. One is from Property 3, and the other is using "equal power allocation" to approximate "water-filling power allocation." To evaluate the accuracy of the two approximations, we show the accurate achievable rates per data stream and their bounds in Fig. 1. In the legends, "Equal PA" and "Water-filling PA" denote the simulation results of achievable rates with equal and water-filling power allocations, respectively. The legends "Upper bound" and "Lower bound" denote the numerical results from the arithmetic mean and the geometric mean of the eigenvalues of the channel correlation matrix according to Property 3.

As shown in Fig. 1, for both FDM multicarrier systems and MIMO systems, no matter whether we consider equal or water-filling power allocations, the lower bound is close to the simulation results. Compared with the upper bound obtained from the Jensen's inequality, the proposed lower bound is much tighter, particularly at higher SNR levels. Fig. 2 shows the achievable rate per data stream versus the order of generalized mean p of multicarrier relay systems.

We then verify that the derived expressions of generalized means with different orders are consistent with the simulations. In Fig. 2, we compare the simulation results of the achievable rate after two hops in the considered cases in Table II with the numerical results obtained from the generalized means for the multicarrier system with M subcarriers. To show the results of the cases of "full diversity" and "full multiplexing" in one figure, we show the achievable rate per data stream, which is in fact a normalized end-to-end achievable rate over the number of data streams. We also provide the performance of the single subcarrier system, i.e., $M = 1$, as a baseline. For the cases ①, ②, ③ and ④, the results obtained from the generalized means are the accurate achievable rates, the simulation results overlap with the numerical results as expected. For the cases ⑤ and ⑥, the results obtained from the generalized means are the lower bounds of the achievable rates, where the simulation

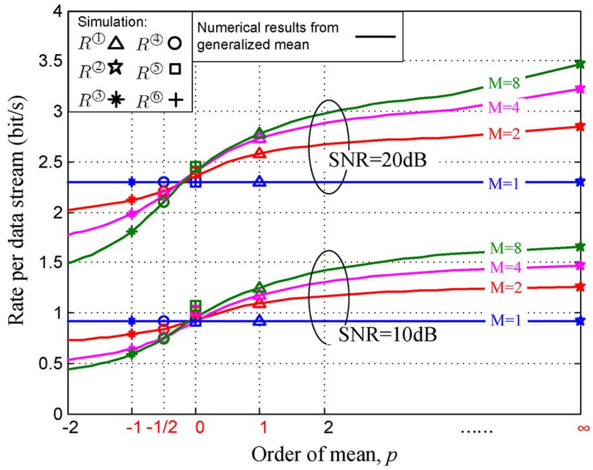

Fig. 2. Achievable rate per data stream versus the order of generalized mean p for multicarrier relay systems.

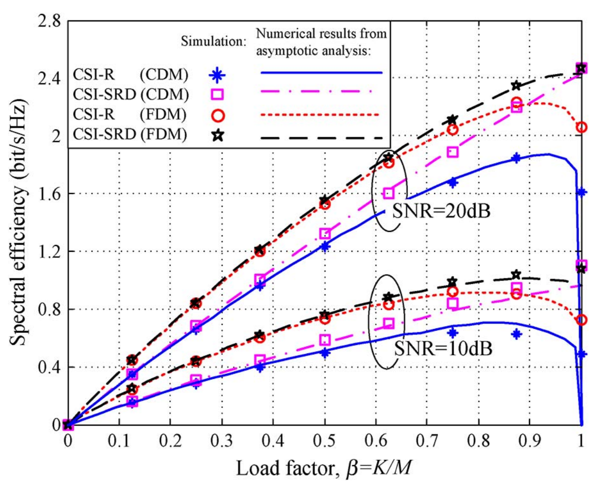

Fig. 3. Spectral efficiency versus the load factor β for multicarrier relay systems.

results consider the corresponding optimal power allocations. We can see that the simulation results are also quite close to the numerical results, which shows the tightness of the lower bounds, even when the SNR is not high. Similar results can be obtained for the MIMO relay system, which are not shown due to space limitations.

In Figs. 3 and 4, we show the spectral efficiency of multicarrier and MIMO two-hop relay systems, respectively. In Fig. 3, we compare the simulation results obtained from $M = 32$ subcarriers with the numerically computed asymptotic spectral efficiency values from (22a)–(22d). In Fig. 4, we compare the simulation results for an MIMO system with $M = 8$ antennas with the asymptotic analysis results obtained from (22e) and (22f). It shows that the numerical results are close to the simulation results with finite numbers of K and M for different SNRs. This implies that the asymptotic spectral efficiency converges to the average spectral efficiency rapidly.

From both the analytical and simulation results, we can obtain the following observations: 1) The maximal spectral efficiency is provided by full multiplexing (i.e., $\beta = 1$) only when the CSI is available at all three nodes; 2) the performance gain provided by gathering all CSI at the three nodes

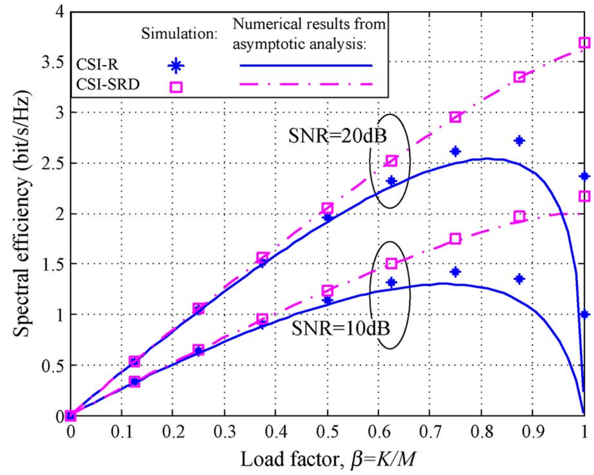

Fig. 4. Spectral efficiency versus the load factor β for MIMO relay systems.

increases with the load factor β; and 3) the performance gain depends on the employed resources and the particular transmission strategy. This can be observed by comparing the maximal asymptotic spectral efficiency at a high SNR level (e.g., SNR $=$ 20 dB). For the case of CSI-R, the maximal spectral efficiency of multicarrier relay systems using CDM with orthogonal spreading sequences and using FDM are 1.87 ($\beta = 0.94$) and 2.22 bit/s/Hz ($\beta = 0.91$), respectively, and that of MIMO relay systems is 2.54 bit/s/Hz ($\beta = 0.81$). For the case of CSI-SRD, the maximal spectral efficiency values of multicarrier relay systems using CDM and FDM are all equal to 2.44 bit/s/Hz ($\beta = 1$), and that of MIMO relay systems becomes 3.60 bit/s/Hz ($\beta = 1$). It indicates that the systems where all nodes have CSI can improve, respectively, by 30% (for multicarrier CDM systems), 10% (for multicarrier FDM systems), and 42 % (for MIMO systems) in maximal spectral efficiency over the systems where only the relay node has CSI. Therefore, we can conclude that, for multicarrier relay system using CDM with orthogonal spreading sequences and for MIMO relay system, gathering the CSI at all three nodes provides substantial performance gain. In contrast, for OFDM relay systems, the performance gain is marginal.

VI. Conclusion

In this paper, we have employed the generalized mean to unify the expressions of achievable rates of multicarrier and multiantenna two-hop DF relay systems with different channel knowledge. Although some of the expressions are lower bounds of the achievable rates at general SNR level, simulations show that the bounds are very tight. By further using random matrix theory, we have obtained corresponding closed-form asymptotic spectral efficiency values. With the unified analytical framework, it is convenient to analyze the impact of available channel information on the relay systems with different settings. It showed from both analytical analysis and simulation results that the performance gain provided by gathering channels at all nodes is increasing with the load factor and the gain depends on the spatial or frequency resources as well as the transmit strategy. For OFDM relay systems, the performance gain is very limited. For multicarrier relay system using CDM with orthogonal spreading sequences and MIMO relay system, the performance gain is remarkable.

APPENDIX
PROOF OF PROPERTY 3

From Jensen's inequality, it is easy to obtain that $\log(1 + \mathcal{M}_1(x_1, \ldots, x_M)) \geq \sum_{m=1}^{M} \log(1 + x_m)/M$. In the following, we only need to prove

$$\frac{1}{M}\sum_{m=1}^{M}\log(1+x_m) \geq \log\left(1+\mathcal{M}_0(x_1,\ldots,x_M)\right). \quad (54)$$

The left-hand side of (54) can be rewritten as

$$\begin{aligned}\frac{1}{M}\log\left(\prod_{m=1}^{M}(1+x_m)\right) &= \frac{1}{M}\log\Bigg(1+\sum_{m_1=1}^{M}x_{m_1}+\sum_{m_1=1}^{M}\sum_{m_2=m_1+1}^{M}x_{m_1}x_{m_2}\\ &\quad +\cdots+\sum_{m_1=1}^{M}\cdots\sum_{m_M=m_{M-1}+1}^{M}x_{m_1}\cdots x_{m_M}\Bigg)\\ &= \frac{1}{M}\log\left(1+\sum_{n=1}^{M}s_n\right) \end{aligned} \quad (55)$$

where $s_n = \sum_{m_1=1}^{M}\cdots\sum_{m_n=m_{n-1}+1}^{M} x_{m_1}\cdots x_{m_n}$.

To simplify the expression of s_n, we define a sequence $\mathcal{X}_n \triangleq \{\Pi_{i=1}^{n} x_{m_i} | 1 \leq m_1 < m_2 < \cdots < m_n \leq M\}, n = 1, \ldots, M$. Then, s_n can be expressed as the sum of the entries in $\mathcal{X}_n$. For example, when $M = 3$ and $n = 2$, $\mathcal{X}_n = \{x_1x_2, x_1x_3, x_2x_3\}$ and $s_n = x_1x_2 + x_1x_3 + x_2x_3$.

From the definition of $\mathcal{X}_n$, we know that each entry of $\mathcal{X}_n$ is a product of n different entries in $\{x_1, \ldots, x_n\}$. Therefore, we know that there are $\binom{M}{n}$ entries in $\mathcal{X}_n$. Therefore, s_n can be expressed as $s_n = \binom{M}{n}\mathcal{M}_1(\mathcal{X}_n)$.

Let X_{n_i} be the ith entry of $\mathcal{X}_n$, from Property 2, we have

$$\mathcal{M}_1(\mathcal{X}_n) \geq \mathcal{M}_0(\mathcal{X}_n) = \sqrt[\binom{M}{n}]{\prod_{i=1}^{\binom{M}{n}} X_{n_i}}. \quad (56)$$

According to the definition of $\mathcal{X}_n$, the entries in $\mathcal{X}_n$ that include x_1 can be expressed as $x_1 \cdot (x_{m_2} \cdots x_{m_n})$, where $1 < m_2 < \cdots < m_n \leq M$. Since $x_{m_2}, \ldots, x_{m_n}$ is a product of $n-1$ different entries in $\{x_2, \ldots, x_n\}$, there are $\binom{M-1}{n-1}$ entries in $\mathcal{X}_n$ that include x_1. Therefore, $\Pi_{i=1}^{\binom{M}{n}} X_{n_i}$ includes the term $x_1^{\binom{M-1}{n-1}}$. Similarly, we can show that $\Pi_{i=1}^{\binom{M}{n}} X_{n_i}$ consist of the term $x_j^{\binom{M-1}{n-1}}$, $j = 2, \ldots, M$. Then, we have

$$\prod_{i=1}^{\binom{M}{n}} X_{n_i} = (x_1 \ldots x_M)^{\binom{M-1}{n-1}} = \bar{x}^{M\binom{M-1}{n-1}} \quad (57)$$

where $\bar{x} = \mathcal{M}_0(x_1, \ldots, x_M) = \sqrt[M]{x_1, \ldots, x_M}$.

Substituting (57) into (56), we have

$$\mathcal{M}_1(\mathcal{X}_n) \geq \bar{x}^{\frac{M\binom{M-1}{n-1}}{\binom{M}{n}}} = \bar{x}^{\frac{M(M-1)!}{(n-1)!(M-n)!}\frac{n!(M-n)!}{M!}} = \bar{x}^n. \quad (58)$$

Then, from (55), we obtain

$$\begin{aligned}\frac{1}{M}\log\left(\prod_{m=1}^{M}(1+x_m)\right) &\geq \frac{1}{M}\log\left(\sum_{n=0}^{M}\binom{M}{n}\bar{x}^n\right)\\ &= \frac{1}{M}\log\left((1+\bar{x})^M\right)\\ &= \log\left(1+\mathcal{M}_0\left(x_1,\ldots,x_M\right)\right)\end{aligned} \quad (59)$$

i.e., (54). Therefore, we obtain (11).

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